Introductory Algebra

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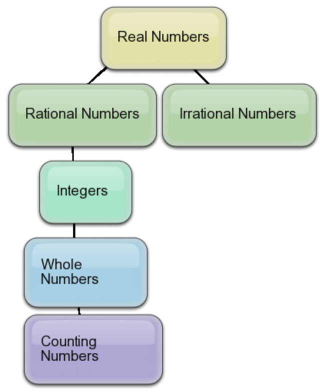
CHAPTER 1 Arithmetic Review

Chapter Outline

Real numbers are all around us. The majority of numbers we use in calculations are considered **real numbers.** This chapter defines a real number and explains important properties and rules that apply to real numbers.

1.1 Integers and Rational Numbers

Integers and rational numbers are important in daily life. The price per square yard of carpet is a rational number. The number of frogs in a pond is expressed using an integer. The organization of real numbers can be drawn as a **hierarchy**. Look at the hierarchy below.



The most generic number is the real number; it can be a combination of negative, positive, decimal, fraction, or non-repeating decimal values. Real numbers have two major categories: rational numbers and irrational numbers.

Definition: Irrational numbers are numbers that can be written as non-repeating, non-terminating decimals such as π or $\sqrt{2}$.

Definition: Rational numbers are numbers that can be written in the form $\frac{a}{b}$, where a and b are integers *and* $b \neq 0$.

All Irrational Numbers and Rational Numbers are Real Numbers

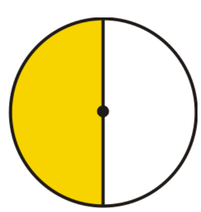
Definition: Integers are all the whole numbers, zero and the negatives of the whole numbers. i.e. {. . . -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5. . . } All Integers are Rational Numbers.

Definition: Whole Numbers are all Counting Numbers and the number zero. i.e. {0, 1, 2, 3, 4, 5. . . } All Whole Numbers are Integer.

Definition: Counting Numbers are the the natural numbers from 1 to infinity. i.e. {1, 2, 3, 4, 5. . . } All Counting Numbers are Whole Numbers.

A Review of Fractions

You can think of a rational number as a fraction of a cake. If you cut the cake into b slices, your share is a of those slices. For example, when we see the rational number $\frac{1}{2}$, we imagine cutting the cake into two parts. Our share is one of those parts. Visually, the rational number $\frac{1}{2}$ looks like this.

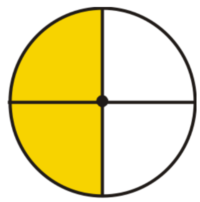


There are two main types of fractions:

- **Proper fractions** are rational numbers where the numerator is less than the denominator. A proper fraction represents a number less than one. With a proper fraction you always end up with less than a whole cake!

- **Improper fractions** are rational numbers where the numerator is greater than or equal to the denominator. Improper fractions can be rewritten as a mixed number – an integer plus a proper fraction. An improper fraction represents a number greater than or equal to one.

When evaluating fractions, it is possible for two fractions to give the same numerical value. These fractions are called **equivalent fractions**. For example, look at a visual representation of the rational number $\frac{2}{4}$.



The visual of $\frac{1}{2}$ is equivalent to the visual of $\frac{2}{4}$. We can write out the prime factors of both the numerator and the denominator and cancel matching factors that appear in both the numerator **and** denominator.

$\left(\frac{2}{4}\right) = \left(\frac{\cancel{2} \cdot 1}{\cancel{2} \cdot 2 \cdot 1}\right)$ We then re-multiply the remaining factors. $\left(\frac{2}{4}\right) = \left(\frac{1}{2}\right)$

Therefore, $\frac{1}{2} = \frac{2}{4}$. This process is called **reducing** the fraction, or writing the fraction in lowest terms. Reducing a fraction does not change the value of the fraction; it simplifies the way we write it. When we have canceled all common factors, we have a fraction in its **simplest form**.

Example 1: *Classify and simplify the following rational numbers.*

a) $\left(\frac{3}{7}\right)$

b) $\left(\frac{9}{3}\right)$

c) $\left(\frac{50}{60}\right)$

Solution:

a) Because both 3 and 7 are prime numbers, $\frac{3}{7}$ is a proper fraction written in its simplest form.

b) The numerator is larger than the denominator, therefore, this is an improper fraction.

$$\frac{9}{3} = \frac{3 \times 3}{3} = \frac{3}{1} = 3$$

c) This is a proper fraction; $\frac{50}{60} = \frac{5 \times 2 \times 5}{6 \times 2 \times 5} = \frac{5}{6}$

Ordering Rational Numbers

To order rational numbers is to arrange them according to a set of directions, such as ascending (lowest to highest) or descending (highest to lowest). Ordering rational numbers is useful when determining which unit cost is the cheapest.

Example 2: *Cans of tomato sauce come in three sizes: 8 ounces, 16 ounces, and 32 ounces. The costs for each size are \$0.59, \$0.99, and \$1.29, respectively. Find the unit cost and order the rational numbers in ascending order.*

Solution: Use proportions to find the cost per ounce. $\frac{\$0.59}{8} = \frac{\$0.07375}{ounce}$; $\frac{\$0.99}{16} = \frac{\$0.061875}{ounce}$; $\frac{\$1.29}{32} = \frac{\$0.040}{ounce}$. Arranging the rational numbers in ascending order: 0.040, 0.061875, 0.07375

Example 3: *Which is greater* $\frac{3}{7}$ *or* $\frac{4}{9}$?

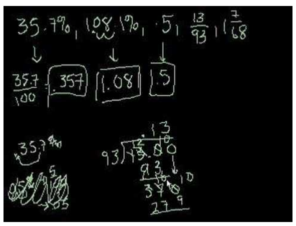
Solution: Begin by creating a common denominator for these two fractions. Which number is evenly divisible by 7 and 9? $7 \times 9 = 63$, therefore the common denominator is 63.

$$\frac{3 \times 9}{7 \times 9} = \frac{27}{63} \qquad \frac{4 \times 7}{9 \times 7} = \frac{28}{63}$$

Because $28 > 27$, $\frac{4}{9} > \frac{3}{7}$

For more information regarding how to order fractions, watch this YouTube video.

KhanAcademy: Ordering Fractions



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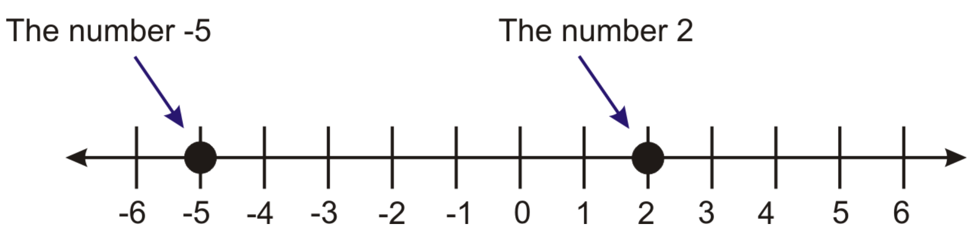
Graph and Compare Integers

More specific than the rational numbers are the **integers.** Integers are whole numbers and their negatives. When comparing integers, you will use the math verbs such as less than, greater than, approximately equal to, and equal to. To graph an integer on a number line, place a dot above the number you want to represent.

Example 4: *Compare the numbers 2 and -5.*

Solution: First, we will plot the two numbers on a number line.



We can compare integers by noting which is the **greatest** and which is the **least.** The **greatest** number is farthest to the right, and the **least** is farthest to the left.

In the diagram above, we can see that 2 is farther to the right on the number line than -5, so we say that **2 is greater than -5**. We use the symbol >to mean "greater than".

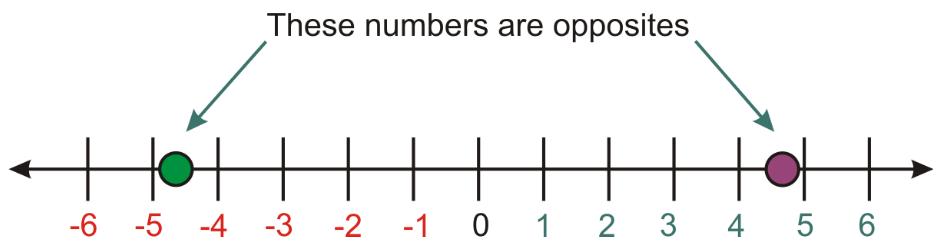
Therefore, $2 > -5$.

This can also be read as **-5 is less than 2**, because the -5 is farther to the left on the number line. To write this, we use the symbol <to mean "less than".

Therefore, $-5 < 2$

Numbers and Their Opposites

Every number has an **opposite,** which represents the same distance from zero but in the other direction.



A special situation arises when adding a number to its opposite. The sum is zero. This is summarized in the following property.

The **Additive Inverse Property:** For any real number a, $a + -a = 0$.

Absolute Value

Absolute value represents the distance from zero when graphed on a number line. For example, the number 7 is 7 units away from zero. The number -7 is also 7 units away from zero. Therefore, the absolute value of 7 and the absolute value of -7 are both 7.

We **write** the absolute value of -7 like this: $|-7|$.

We **read** the expression $|x|$ like this: "the absolute value of x."

- Treat absolute value expressions like parentheses. If there is an operation inside the absolute value symbols evaluate that operation first.
- The absolute value of a number or an expression is **always** positive or zero. It cannot be negative. With absolute value, we are only interested in how far a number is from zero, not the direction.

Example 5: *Evaluate the following absolute value expressions.*

a) $|5+4|$

b) $3-|4-9|$

c) $|-5-11|$

d) $-|7-22|$

Solution:

a)

$$\begin{aligned} |5+4| &= |9| \\ &= 9 \end{aligned}$$

b)

$$\begin{aligned} 3-|4-9| &= 3-|-5| \\ &= 3-5 \\ &= -2 \end{aligned}$$

c)

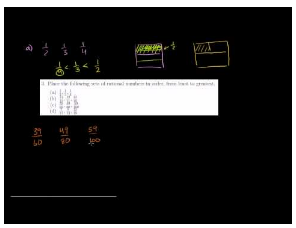
$$\begin{aligned} |-5-11| &= |-16| \\ &= 16 \end{aligned}$$

d)

$$\begin{aligned} -|7-22| &= -|-15| \\ &= -(15) \\ &= -15 \end{aligned}$$

Practice Set: Integers and Rational Numbers

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. Integersand Rational Numbers (13:00)



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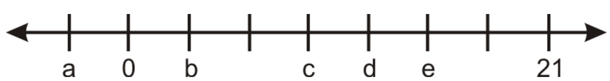
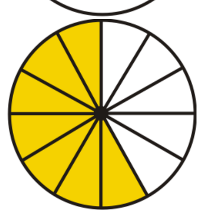
1. Define absolute value.
2. What are the two types of fractions?
3. Give an example of a real number that is not an integer.
4. What standards separate a rational number from an irrational number?
5. The tick-marks on the number line represent evenly spaced integers. Find the values of a, b, c, d and e.

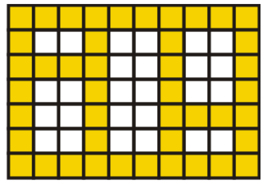


In 6 – 8, determine what fraction of the whole each shaded region represents.

6.



7.

8. 

In 9 – 12, place the following sets of rational numbers in order from least to greatest.

9. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$
10. $\frac{11}{12}, \frac{12}{11}, \frac{13}{10}$
11. $\frac{39}{60}, \frac{49}{80}, \frac{59}{100}$
12. $\frac{7}{11}, \frac{8}{13}, \frac{12}{19}$

In 13 – 18, find the simplest form of the following rational numbers.

13. $\frac{22}{44}$
14. $\frac{9}{27}$
15. $\frac{12}{18}$
16. $\frac{315}{420}$
17. $\frac{19}{101}$
18. $\frac{99}{11}$

In 19 – 27, simplify.

19. $|-98.4|$

20. $|123.567|$

21. $-|16-98|$

22. $11-|-4|$

23. $|4-9|-|-5|$

24. $|-5-11|$

25. $-|-7|$

26. $|-2-88|-|88+2|$

27. $|-5-99|-|16-7|$

In 28 – 33, compare the two real numbers using the symbol <or >.

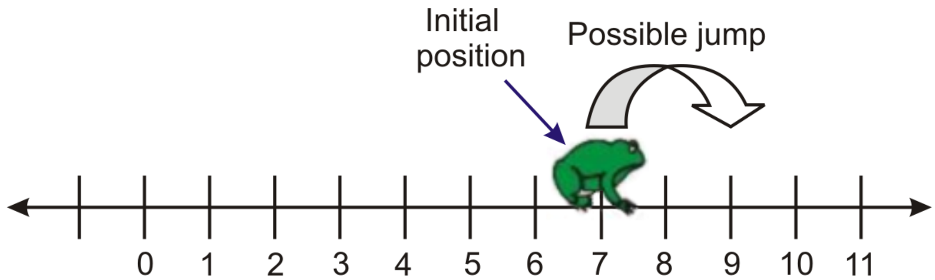
28. 8 and 7.99999

29. -4.25 and $\frac{-17}{4}$

30. 65 and -1

31. 10 units left of zero and 9 units right of zero

32. *A frog is sitting perfectly on top of number 7 on a number line. The frog jumps randomly to the left or right, but always jumps a distance of exactly 2. Describe the set of numbers that the frog may land on, and list all the possibilities for the frog's position after exactly 5 jumps.*



33. Will a real number always have an additive inverse? Explain your reasoning.

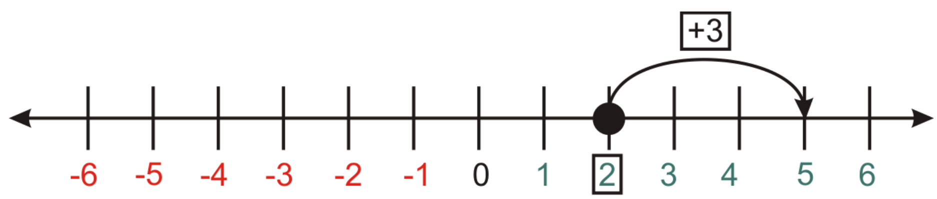
1.2 Addition and Subtraction of Rational Numbers

Addition of Rational Numbers

A football team gains 11 yards on one play then loses 5 yards on another play and loses 2 yards on the third play. What is the total yardage loss or gain?

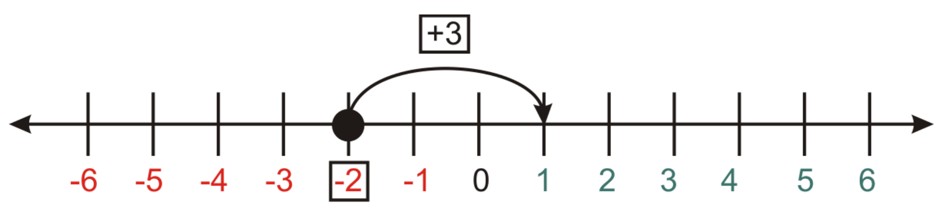
A loss can be expressed as a negative integer. A gain can be expressed as a positive integer. To find the net gain or loss, the individual values must be added together. Therefore, the sum is $11 + -5 + -2 = 4$. The team has a net gain of 4 yards.

Addition can also be shown using a number line. If you need to add $2+3$, start by making a point at the value of 2 and move three integers to the right. The ending value represents the sum of the values.



Example 1: *Find the sum of* $-2+3$ *using a number line.*

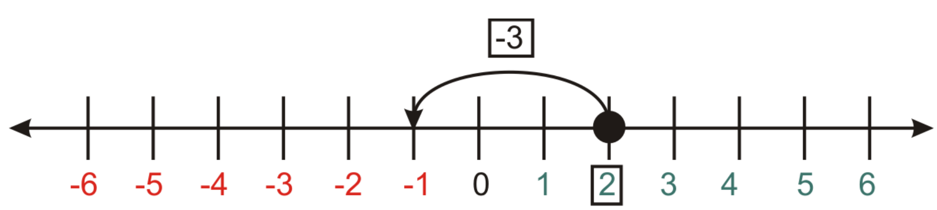
Solution: Begin by making a point at -2 and moving three units to the right. The final value is 1, so $-2+3=1$



When the value that is being added is positive, we "jump" to the right. If the value is negative, we jump to the left (in a negative direction).

Example 2: *Find* $2-3$ *using a number line.*

Solution: Begin by making a point at 2. The expression represents subtraction, so we will count three jumps to the left.



The solution is: $2-3=-1$

Algebraic Properties of Addition

In a previous lesson you learned the Additive Inverse Property. This property states that the sum of a number and its opposite is zero. Algebra has many other properties that help you manipulate and organize information.

The **Commutative Property of Addition:** For all real numbers a, and b, $a+b=b+a$.

To *commute* means to change locations, so the Commutative Property of Addition allows you to rearrange the objects in an addition problem.

The **Associative Property of Addition:** For all real numbers $a,\ b$, and c, $(a+b)+c=a+(b+c)$

To *associate* means to group together, so the Associative Property of Addition allows you to regroup the objects in an addition problem.

The **Identity Property of Addition:** For any real number $a,\ a+0=a$

This property allows you to use the fact that the sum of any number and zero is the original value. For this reason, we call zero the additive identity.

Example 3: *Simplify the following using the properties of addition:*

a) $9+(1+22)$

b) $4211+0$

Solution:

a) It is easier to regroup $9+1$, so by applying the Associative Property of Addition, $(9+1)+22=10+22=32$

b) The Additive Identity Property states the sum of a number and zero is itself, therefore $4211+0=4211$

Nadia and Peter are building sand castles on the beach. Nadia built a castle two feet tall, stopped for ice-cream and then added one more foot to her castle. Peter built a castle one foot tall before stopping for a sandwich. After his sandwich, he built up his castle by two more feet. Whose castle is the taller?

Nadia's castle is $(2+1)$ feet tall. Peter's castle is $(1+2)$ feet tall. According to the **Commutative Property of Addition,** the two castles are the same height.

Adding Rational Numbers

To add rational numbers, we must first remember how to rewrite mixed numbers as improper fractions. Begin by multiplying the denominator of the mixed number to the whole value. Add the numerator to this product. This value is the numerator of the improper fraction. The denominator is the original.

Example 4: *Write* $11\frac{2}{3}$ as an improper fraction:

Solution: $3 \times 11 = 33 + 2 = 35$. This is the numerator of the improper fraction.

$$11\frac{2}{3} = \frac{35}{3}$$

Now that we know how to rewrite a mixed number as an improper fraction, we can begin to add rational numbers. There is one thing to remember when finding the sum or difference of rational numbers: the denominators must be equivalent.

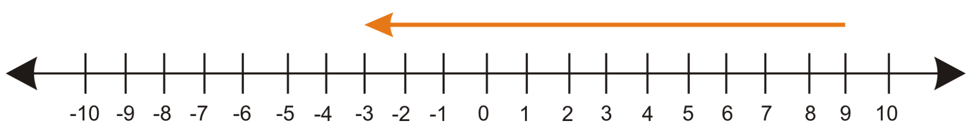
The **Addition Property of Fractions:** For all real numbers a, b, and c, $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$

Watch this video for further explanation on adding fractions with unlike denominators. This video shows how to add fractions using a visual model.

http://www.teachertube.com/viewVideo.php?video_id=103926&title=Adding_Fractions_with_Unlike_Denominators

Subtraction of Rational Numbers

In the previous two lessons you have learned how to find the opposite of a rational number and to add rational numbers. You can use these two concepts to subtract rational numbers. Suppose you want to find the difference of 9 and 12. Symbolically, it would be $9 - 12$. Begin by placing a dot at nine and move to the **left** 12 units.



$$9 - 12 = -3$$

Rule: To subtract a number, add its opposite.

$$3 - 5 = 3 + (-5) = -2 \qquad 9 - 16 = 9 + (-16) = -7$$

A special case of this rule can be written when trying to subtract a negative number.

The **Opposite-Opposite Property:** Since taking the opposite of a number changes its sign, we can say that $-(-b) = b$. So it is also true that for any real numbers a and b, $a - (-b) = a + b$.

Example 1: *Simplify* $-6 - (-13)$

Solution: Using the Opposite-Opposite Property, the double negative is rewritten as a positive.

$$-6 - (-13) = -6 + 13 = 7$$

Example 2: *Simplify* $\frac{5}{6} - \left(-\frac{1}{18}\right)$:

Solution: Begin by using the Opposite-Opposite Property

$$\frac{5}{6} + \frac{1}{18}$$

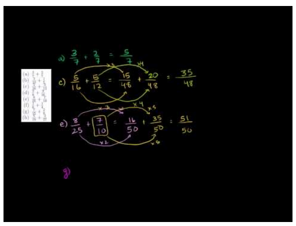
Next, create a common denominator: $\frac{5\times 3}{6\times 3}+\frac{1}{18}=\frac{15}{18}+\frac{1}{18}$

Add the fractions: $\frac{16}{18}$

Reduce: $\frac{2\times 2\times 2\times 2}{3\times 3\times 2}=\frac{8}{9}$

Practice Set: Addition of Integers

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. Additionof RationalNumbers (7:40)



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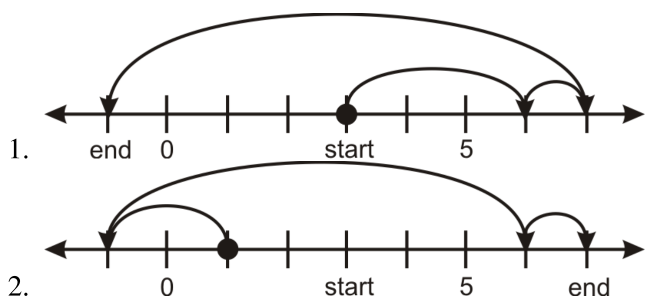
In exercises 1 and 2, write the sum represented by the moves on the number line.



Find the sum. Write fractions in their **simplest form.** Convert improper fractions to mixed numbers.

3. $\frac{3}{7}+\frac{2}{7}$
4. $\frac{3}{10}+\frac{1}{5}$
5. $\frac{5}{16}+\frac{5}{12}$
6. $\frac{3}{8}+\frac{9}{16}$
7. $\frac{8}{25}+\frac{7}{10}$
8. $\frac{1}{6}+\frac{1}{4}$
9. $\frac{7}{15}+\frac{2}{9}$
10. $\frac{5}{19}+\frac{2}{27}$
11. $-2.6+11.19$
12. $-8+13$
13. $-7.1+(-5.63)$
14. $9.99+(-0.01)$

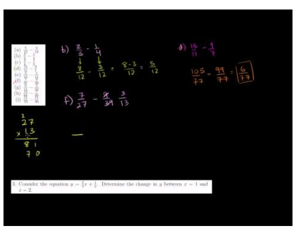
15. $4\frac{7}{8}+1\frac{1}{2}$
16. $-3\frac{1}{3}+\left(-2\frac{3}{4}\right)$

In 17 – 20, which property of addition does each situation involve?

17. Whichever order your groceries are scanned at the store, the total will be the same.
18. Suppose you go buy a DVD for \$8.00, another for \$29.99, and a third for \$14.99. You can add $(8+29.99)+14.99$ or you can add $8+(29.99+14.99)$ to obtain the total.
19. Shari's age minus the negative of Jerry's age equals the sum of the two ages.
20. Kerri has 16 apples and has added zero additional apples. Her current total is 16 apples.
21. Nadia, Peter and Ian are pooling their money to buy a gallon of ice cream. Nadia is the oldest and gets the greatest allowance. She contributes half of the cost. Ian is next oldest and contributes one third of the cost. Peter, the youngest, gets the smallest allowance and contributes one fourth of the cost. They figure that this will be enough money. When they get to the check-out, they realize that they forgot about sales tax and worry there will not be enough money. Amazingly, they have exactly the right amount of money. What fraction of the cost of the ice cream was added as tax?
22. A blue whale dives 160 feet below the surface then rises 8 feet. Write the addition problem and find the sum.
23. The temperature in Chicago, Illinois one morning was $-8^{\circ}F$. Over the next six hours the temperature rose 25 degrees Fahrenheit. What was the new temperature?

Practice Set: Subtraction of Integers

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. Subtraction of Rational Numbers (10:22)



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In 1 – 20, subtract the following rational numbers. Be sure that your answer is in the **simplest form.**

1. $9-14$
2. $2-7$
3. $21-8$
4. $8-(-14)$
5. $-11-(-50)$
6. $\frac{5}{12}-\frac{9}{18}$
7. $5.4-1.01$
8. $\frac{2}{3}-\frac{1}{4}$
9. $\frac{3}{4}-\frac{1}{3}$
10. $\frac{1}{4}-\left(-\frac{2}{3}\right)$
11. $\frac{15}{11}-\frac{9}{7}$
12. $\frac{2}{13}-\frac{1}{11}$
13. $-\frac{7}{8}-\left(-\frac{8}{3}\right)$
14. $\frac{7}{27}-\frac{9}{39}$
15. $\frac{6}{11}-\frac{3}{22}$
16. $-3.1-21.49$
17. $\frac{13}{64}-\frac{7}{40}$
18. $\frac{11}{70}-\frac{11}{30}$
19. $-68-(-22)$
20. $\frac{1}{3}-\frac{1}{2}$
21. KMN stock began the day with a price of \$4.83 per share. At the closing bell, the price dropped \$0.97 per share. What was the closing price of KMN stock?

1.3 Multiplication and Division of Rational Numbers

When you began learning how to multiply whole numbers, you replaced repeated addition with the multiplication sign $(\times)$. For example,

$$6+6+6+6+6=5\times 6=30$$

Multiplying rational numbers is performed the same way. We will start with the Multiplication Property of -1.

Properties of Multiplication

The **Multiplication Property of -1:** For any real number $a, (-1)\times a=-a$.

This can be summarized by saying "a number times a negative 1 is the opposite of the number".

Example 1: *Evaluate* $(-1)\cdot 9,876$.

Solution: Using the Multiplication Property of -1: $(-1)\cdot 9,876=-9,876$.

This property can also be used when the values are negative, as shown in example 2.

Example 2: *Evaluate* $(-1)\cdot -322$.

Solution: Using the Multiplication Property of -1: $(-1)\cdot -322=322$.

A basic algebraic property is the Multiplicative Identity. Similar to the Additive Identity, this property states that any value multiplied by 1 will result in the original value.

The **Identity Property of Multiplication:** For any real numbers a, $(1)\times a=a$.

A third property of multiplication is the Multiplication Property of Zero. This property states that any value multiplied by zero will result in zero.

The **Zero Property of Multiplication:** For any real numbers a, $(0)\times a=0$.

Multiplying Rational Numbers

You've decided to make cookies for a party. The recipe you've chosen makes 6 dozen cookies, but you only need 2 dozen. How do you reduce the recipe?

In this case, you should not use subtraction to find the new values. Subtraction means to make less by taking away. You haven't made any cookies, therefore cannot take any away. Instead, you need to make $\frac{2}{6}$ or $\frac{1}{3}$ of the original

recipe. This process involves multiplying fractions.

For any real numbers a, b, c, d where $b \neq 0$ and $d \neq 0$,

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Example 3: *The original cookie recipe calls for 8 cups flour. How much is needed for the reduced recipe?*

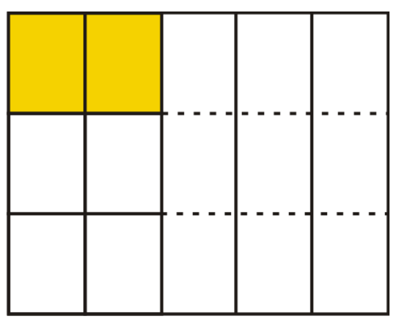
Solution: Begin by writing the multiplication situation. $8 \cdot \frac{1}{3}$. We need to rewrite this product in the form of the property above. In order to perform this multiplication, you need to rewrite 8 as the fraction $\frac{8}{1}$.

$$8 \times \frac{1}{3} = \frac{8}{1} \times \frac{1}{3} = \frac{8 \cdot 1}{1 \cdot 3} = \frac{8}{3} = 2\frac{2}{3}$$

You will need $2\ \frac{2}{3}$ cups flour.

Multiplication of fractions can also be shown visually. For example, to multiply $\frac{1}{3} \cdot \frac{2}{5}$, draw one model to represent the first fraction and a second model to represent the second fraction.

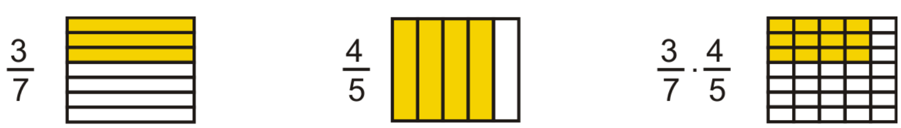
By placing one model (divided in thirds horizontally) on top of the other (divided in fifths vertically) you divide one whole rectangle into *bd* smaller parts. Shade *ac* smaller regions.



The product of the two fractions is the $\frac{shaded\ regions}{total\ regions}$

$$\frac{1}{3} \cdot \frac{2}{5} = \frac{2}{15}$$

Example 4: *Simplify* $\frac{3}{7} \cdot \frac{4}{5}$



Solution: By drawing visual representations, you can see

$$\frac{3}{7} \cdot \frac{4}{5} = \frac{12}{35}$$

More Properties of Multiplication

Properties that hold true for addition such as the Associative Property and Commutative Property also hold true for multiplication. They are summarized below.

The **Associative Property of Multiplication:** For any real numbers a, b, and c,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

The **Commutative Property of Multiplication:** For any real numbers *a and b*,

$$a(b) = b(a)$$

The **Same Sign Multiplication Rule:** The product of two positive or two negative numbers is positive.

The **Different Sign Multiplication Rule:** The product of a positive number and a negative number is a negative number.

Solving Real-World Problems Using Multiplication

Example 5: *Anne has a bar of chocolate and she offers Bill a piece. Bill quickly breaks off* $\frac{1}{4}$ *of the bar and eats it. Another friend, Cindy, takes* $\frac{1}{3}$ *of what was left. Anne splits the remaining candy bar into two equal pieces which she shares with a third friend, Dora. How much of the candy bar does each person get?*

Solution: Think of the bar as one whole.

$1 - \frac{1}{4} = \frac{3}{4}$. This is the amount remaining after Bill takes his piece.

$\frac{1}{3} \times \frac{3}{4} = \frac{1}{4}$. This is the fraction Cindy receives.

$\frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$. This is the amount remaining after Cindy takes her piece.

Anne divides the remaining bar into two equal pieces. Every person receives $\frac{1}{4}$ of the bar.

Example 6: *Doris' truck gets* $10\frac{2}{3}$ *miles per gallon. Her tank is empty so she fills it with* $5\frac{1}{2}$ *gallons.*

How far can she travel?

Solution: Begin by writing each mixed number as an improper fraction.

$$10\frac{2}{3} = \frac{32}{3} \qquad 5\frac{1}{2} = \frac{11}{2}$$

Now multiply the two values together.

$$\frac{32}{3} \cdot \frac{11}{2} = \frac{352}{6} = 58\frac{4}{6} = 58\frac{2}{3}$$

Doris can travel 58 $\frac{2}{3}$ miles on 5 $\frac{1}{2}$ gallons of gas.

Division of Rational Numbers

So far in this chapter you have added, subtracted, and multiplied rational numbers. It now makes sense to learn how to divide rational numbers. We will begin with a definition of **inverse operations.**

Inverse operations "undo" each other.

For example, addition and subtraction are inverse operations because addition cancels subtraction and vice versa. The additive identity results in a sum of zero. In the same sense multiplication and division are inverse operations. This leads into the next property: The Inverse Property of Multiplication.

For every nonzero number a, there is a multiplicative inverse $\frac{1}{a}$ such that $a\left(\frac{1}{a}\right) = 1$.

The values of a and $\frac{1}{a}$ are called **reciprocals.** In general, two nonzero numbers whose product is 1 are reciprocals.

Reciprocal: The reciprocal of a nonzero rational number $\frac{a}{b}$ is $\frac{b}{a}$.

Note: The number zero does not have a reciprocal.

Using Reciprocals to Divide Rational Numbers

When dividing rational numbers, use the following rule:

“When dividing rational numbers, multiply by the ’right’ reciprocal.”

In this case, the “right” reciprocal means to take the reciprocal of the fraction on the right-hand side of the division operator.

Example 7: *Simplify* $\frac{2}{9} \div \frac{3}{7}$.

Solution: Begin by multiplying by the “right” reciprocal

$$\frac{2}{9} \times \frac{7}{3} = \frac{14}{27}$$

Example 8: *Simplify* $\frac{7}{3} \div \frac{2}{3}$.

Solution: Begin by multiplying by the “right” reciprocal.

$$\frac{7}{3} \div \frac{2}{3} = \frac{7}{3} \times \frac{3}{2} = \frac{7 \cdot 3}{2 \cdot 3} = \frac{7}{2}$$

Instead of the division symbol $\div$, you may see a large fraction bar. This is seen in the next example.

Example 9: *Simplify* $\frac{\frac{2}{3}}{\frac{7}{8}}$.

Solution: The fraction bar separating $\frac{2}{3}$ and $\frac{7}{8}$ indicates division.

$$\frac{2}{3} \div \frac{7}{8}$$

Simplify as in example 2:

$$\frac{2}{3} \times \frac{8}{7} = \frac{16}{21}$$

Using Reciprocals to Solve Real-World Problems

The need to divide rational numbers is necessary for solving problems in physics, chemistry, and manufacturing. The following example illustrates the need to divide fractions in physics.

Example 10: *Newton's Second Law relates acceleration to the force of an object and its mass:* $a = \frac{F}{m}$. *Suppose* $F = 7\frac{1}{3}$ *and . Find a, the acceleration.*

Solution: Before beginning the division, the mixed number of force must be rewritten as an improper fraction.

Replace the fraction bar with a division symbol and simplify: $a = \frac{22}{3} \div \frac{1}{5}$

$\frac{22}{3} \times \frac{5}{1} = \frac{110}{3} = 36\frac{2}{3}$. Therefore, the acceleration is $36\frac{2}{3}\ m/s^2$

Example 11: *Anne runs a mile and a half in one-quarter hour. What is her speed in miles per hour?*

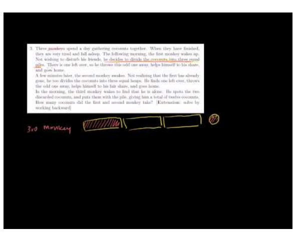
Solution: Use the formula $speed = \frac{distance}{time}$.

$$s = 1.5 \div \frac{1}{4}$$

Rewrite the expression and simplify: $s = \frac{3}{2} \cdot \frac{4}{1} = \frac{4 \cdot 3}{2 \cdot 1} = \frac{12}{2} = 6\ mi/hr$

Practice Set: Multiplication of Integers

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. Multiplicati on of Rational Numbers (8:56)



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Multiply the following rational numbers.

1. $\frac{1}{2} \cdot \frac{3}{4}$
2. $-7.85 \cdot -2.3$

3. $\frac{2}{5} \cdot \frac{5}{9}$
4. $\frac{1}{3} \cdot \frac{2}{7} \cdot \frac{2}{5}$
5. $4.5 \cdot -3$
6. $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5}$
7. $\frac{5}{12} \times \frac{9}{10}$
8. $\frac{27}{5} \cdot 0$
9. $\frac{2}{3} \times \frac{1}{4}$
10. $-11.1(4.1)$
11. $\frac{3}{4} \times \frac{1}{3}$
12. $\frac{15}{11} \times \frac{9}{7}$
13. $\frac{2}{7} \cdot -3.5$
14. $\frac{1}{13} \times \frac{1}{11}$
15. $\frac{7}{27} \times \frac{9}{14}$
16. $\left(\frac{3}{5}\right)^2$
17. $\frac{1}{11} \times \frac{22}{21} \times \frac{7}{10}$
18. $5.75 \cdot 0$

In 19 – 21, state the property that applies to each of the following situations.

19. A gardener is planting vegetables for the coming growing season. He wishes to plant potatoes and has a choice of a single 8 by 7 meter plot, or two smaller plots of 3 by 7 meters and 5 by 7 meters. Which option gives him the largest area for his potatoes?

20. Andrew is counting his money. He puts all his money into $10 piles. He has one pile. How much money does Andrew have?

21. Nadia and Peter are raising money by washing cars. Nadia is charging $3 per car, and she washes five cars in the first morning. Peter charges $5 per car (including a wax). In the first morning, he washes and waxes three cars. Who has raised the most money?

In 22 – 30, find the multiplicative inverse of each of the following.

22. 100

23. $\frac{2}{8}$

24. $-\frac{19}{21}$

25. 7

26. $-\frac{z^3}{2xy^2}$

27. 0

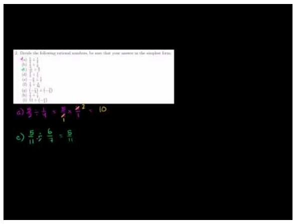
28. $\frac{1}{3}$

29. $\frac{-19}{18}$

30. $\frac{3xy}{8z}$

Practice Set: Division of Integers

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. Divisionof RationalNumbers (8:20)



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In 1 - 9, divide the following rational numbers. Be sure that your answer in the simplest form.

1. $\frac{5}{2} \div \frac{1}{4}$
2. $\frac{1}{2} \div \frac{7}{9}$
3. $\frac{5}{11} \div \frac{6}{7}$
4. $\frac{1}{2} \div \frac{1}{2}$
5. $-\frac{x}{2} \div \frac{5}{7}$
6. $\frac{1}{2} \div \frac{x}{4y}$
7. $\left(-\frac{1}{3}\right) \div \left(-\frac{3}{5}\right)$
8. $\frac{7}{2} \div \frac{7}{4}$
9. $11 \div \left(-\frac{x}{4}\right)$

In 10 – 12, evaluate the expression.

10. $\frac{x}{y}$ *for* $x = \frac{3}{8}$ *and* $y = \frac{4}{3}$

11. $4z \div u$ *for* $u = 0.5$ *and* $z = 10$

12. $\frac{-6}{m}$ *for* $m = \frac{2}{5}$

In 13 - 16, answer the questions.

13. The label on a can of paint states that it will cover 50 square feet per pint. If I buy a $\frac{1}{8}$ pint sample, it will cover a square two feet long by three feet high. Is the coverage I get more, less or the same as that stated on the label?

14. The world's largest trench digger, "Bagger 288," moves at $\frac{3}{8}$ mph. How long will it take to dig a trench $\frac{2}{3}$ mile long?

15. Explain why the reciprocal of a nonzero rational number is not the same as the opposite of that number.

16. Explain why zero does not have a reciprocal.

1.4 Order of Operations

The Aviary Dilemma

Keisha loves the birds in the aviary at the city zoo. Her favorite part of the aviary is the bird rescue. Here the zoo staff rescues injured birds, helps them to heal and then releases them again. Currently, they have 256 birds in the rescue. Today, Keisha has a special visit planned with Ms. Thompson who is in charge of the bird rescue.

When Keisha arrives, Ms. Thompson is already hard at work. She tells Keisha that there are new baby birds in the rescue. Three of the birds have each given birth to five baby birds. Keisha can't help grinning as she walks around. She can hear the babies chirping. In fact, it sounds like they are everywhere.

"It certainly sounds like a lot more babies," Keisha says.

"Yes," Ms. Thompson agrees. "We also released two birds yesterday."

"That is great news," Keisha says smiling.

"Yes, but we also found three new injured birds. Our population has changed again."

"I see," Keisha adds, "That is $256+3\times 5-2+3$ that equals 1296 birds, I think. I'm not sure, that doesn't seem right."

Is Keisha's math correct?

How many birds are there now?

Can you figure it out?

This is a bit of a tricky question. You will need to learn some new skills to help Keisha determine the number of birds in the aviary.

Pay attention. By the end of the lesson, you will know all about the order of operations. Then you will be able to help Keisha with the bird count.

I. **<u>Evaluating Numerical Expressions with the Four Operations</u>**

This lesson begins with evaluating numerical expressions. But before we can do that we need to answer one key question, **"What is an *expression*?"**

To understand what an ***expression*** is, let's compare it with an ***equation***.

An equation is a number sentence that can be solved. It has an equal sign where one side of the equals sign is equal to the other side of the equals sign.

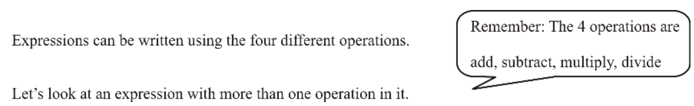
Example

$$3+4=7$$

This is an equation. It has an equals sign and can be solved.

What is an expression then?

An expression is a number sentence without an equals sign. It can be simplified and/or evaluated.



Example

$$4+3\times 5$$

Now this expression can be confusing because it has both addition and multiplication in it.

Do we need to add or multiply first?

To figure this out, we are going to learn something called the *Order of Operations.*

The Order of Operation is a way of evaluating expressions. It lets you know what order to complete each operation in.

<u>Order of Operations</u>

P - parentheses

E - exponents

MD - multiplication or division in order from left to right

AS - addition or subtraction in order from left to right

Take a few minutes to write these down in a notebook.

Now that you know the order of operations, let's go back to our example.

Example

$$4+3\times 5$$

Here we have an expression with addition and multiplication.

We can look at the order of operations and see that multiplication comes before addition. We need to complete that operation first.

$$4+3\times 5$$
$$4+15$$
$$=19$$

When we evaluate this expression using order of operations, our answer is 20.

What would have happened if we had NOT followed the order of operations?

Example

$$4+3\times 5$$

We probably would have solved the problem in order from left to right.

$$4+3\times 5$$
$$7\times 5$$
$$=35$$

This would have given us an incorrect answer. It is important to always follow the order of operations.

Here are few for you to try on your own.

1. $8-1\times 4+3=$ ___
2. $2\times 6+8\div 2=$ ___
3. $5+9\times 3-6+2=$ ___

II. **Evaluating Numerical Expressions Using Powers and Grouping Symbols**

We can also use the order of operations when we have exponent powers and ***grouping symbols*** like parentheses.

Let's review where exponents and parentheses fall in the order of operations.

Order of Operations

P - parentheses

E - exponents

MD - multiplication or division in order from left to right

AS - addition or subtraction in order from left to right

Wow! You can see that according to the order of operations parentheses comes first. We always do the work in parentheses first. Then we evaluate exponents.

Let's see how this works with a new example.

Example

$$2+(3-1)\times 2$$

In this example, we can see that we have four things to look at.

We have 1 set of parentheses, addition, subtraction in the parentheses and multiplication.

We can evaluate this expression using the order of operations.

Example

$$\begin{aligned}&2+(3-1)\times 2\\&2+2\times 2\\&2+4\\&=6\end{aligned}$$

Our answer is 6.

What about when we have parentheses and exponents?

Example

$$35+3^2-(3\times 2)\times 7$$

We start by using the order of operations. It says we evaluate parentheses first.

$$\begin{aligned}&3\times 2=6\\&35+3^2-6\times 7\end{aligned}$$

Next, we evaluate exponents

$$\begin{aligned}&3^2=3\times 3=9\\&35+9-6\times 7\end{aligned}$$

Next, we complete multiplication or division in order from left to right. We have multiplication.

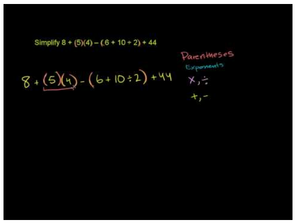
$$\begin{aligned}&6\times 7=42\\&35+9-42\end{aligned}$$

Next, we complete addition and/or subtraction in order from left to right.

$$\begin{aligned}35+9&=44\\44-42&=2\end{aligned}$$

Our answer is 2.

Now, consider the expression $8+(5)(4)-(6+10\div 2)+44$. Simply using order of operations. Then watch the video explanation for this problem.



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Here are a few for you to try on your own.

1. $16+2^3-5+(3\times 4)$
2. $9^2+2^2-5\times(2+3)$
3. $8^2\div 2+4-1\times 6$

III. <u>**Use the Order of Operations to Determine if an Answer is True**</u>

We just finished using the order of operations to evaluate different expressions.

We can also use the order of operations to "check" our work.

In this section, you will get to be a "Math Detective."

As a math detective, you will be using the order of operations to determine whether or not someone else's work is correct.

Here is a worksheet that has been completed by Joaquin.

Your task is to check Joaquin's work and determine whether or not his work is correct.

Use your notebook to take notes.

If the expression has been evaluated correctly, then please make a note of it. If it is incorrect, then re-evaluate the expression correctly.

Here are the problems that are on Joaquin's worksheet.

1. $7\times(4+1)-7\times2=21$
2. $3^2+4^2-9+(3\times2)=22$
3. $6+3\times2-5+(7-1)=19$
4. $(8\times2)-3^2+(5\times2)=17$
5. $18-2\times3+(6\times3)=66$

Did you check Joaquin's work?

Let's see how you did with your answers. Take your notebook and check your work with these correct answers.

Let's begin with problem number 1.

We start by adding $4+1$ which is 5. Then we multiply 7×5 and 7×2. Since multiplication comes next in our order of operations. Finally we subtract $35-14=21$.

Joaquin's work is correct.

Problem Number 2

We start by evaluating the exponents. 3 squared is 9 and 4 squared is 16. Next we multiply $3\times2=6$. Finally we can complete the addition and subtraction in order from left to right. Our final answer is 22.

Joaquin's work is correct.

Problem Number 3

We start with the multiplication and multiply 3×2 which is 6. Then we complete the parentheses $7-1=6$. Now we can complete the addition and subtraction in order from left to right. The answer correct is 13.

Uh Oh, Joaquin's answer is incorrect. How did Joaquin get 19 as an answer?

Well, if you look, Joaquin did not follow the order of operations. He just did the operations in order from left to right. If you don't multiply 3×2 first, then you get 19 as an answer instead of 16.

Problem Number 4

Let's complete the work in parentheses first, $8\times2=16$ and $5\times2=10$. Next we evaluate the exponent, 3 squared is 9. Now we can complete the addition and subtraction in order from left to right. The answer is 17.

Joaquin's work is correct.

Problem Number 5

First, we need to complete the work in parentheses, $6\times3=18$. Next, we complete the multiplication $2\times3=6$. Now we can evaluate the addition and subtraction in order from left to right. Our answer is 30.

Uh Oh, Joaquin got mixed up again. How did he get 66? Let's look at the problem. Oh, Joaquin subtracted $18-2$ before multiplying. You can't do that. He needed to multiply 2×3 first then he needed to subtract. Because of this, Joaquin's work is not accurate.

How did you do?

Remember, a Math Detective can check any answer by following the order of operations.

IV. **Insert Grouping Symbols to Make a Given Answer True**

Sometimes a grouping symbol can help us to make an answer true. By putting a grouping symbol, like parentheses, in the correct spot, we can change an answer.

Let's try this out.

Example

$$5+3\times 2+7-1=22$$

Now if we just solve this problem without parentheses, we get the following answer.

$$5+3\times 2+7-1=17$$

How did we get this answer?

Well, we began by completing the multiplication, $3\times 2=6$. Then we completed the addition and subtraction in order from left to right. That gives us an answer of 17.

However, we want an answer of 22.

Where can we put the parentheses so that our answer is 22?

This can take a little practice and you may have to try more than one spot too.

Let's try to put the parentheses around $5+3$**.**

Example

$$(5+3)\times 2+7-1=22$$

Is this a true statement?

Well, we begin by completing the addition in parentheses, $5+3=8$. Next we complete the multiplication, $8\times 2=16$.

Here is our problem now.

$$16+7-1=22$$

Next, we complete the addition and subtraction in order from left to right.

Our answer is 22.

Here are a few for you to try on your own. Insert a set of parentheses to make each a true statement.

1. $6-3+4\times 2+7=39$
2. $8\times 7+3\times 8-5=65$
3. $2+5\times 2+18-4=28$

The Aviary Dilemma

Let's look back at Keisha and Ms. Thompson and the bird dilemma at the zoo.

Here is the original problem.

Keisha loves the birds in the aviary at the city zoo. Her favorite part of the aviary is the bird rescue. Here the zoo staff rescues injured birds, helps them to heal and then releases them again. Currently, they have 256 birds in the rescue. Today, Keisha is has a special visit planned with Ms. Thompson who is in charge of the bird rescue.

When Keisha arrives, Ms. Thompson is already hard at work. She tells Keisha that there are new baby birds in the rescue. Three of the birds have each given birth to five baby birds. Keisha can't help grinning as she walks around. She can hear the babies chirping. In fact, it sounds like they are everywhere.

"It certainly sounds like a lot more babies," Keisha says.

"Yes," Ms. Thompson agrees. "We also released two birds yesterday."

"That is great news," Keisha says smiling.

"Yes, but we also found three new injured birds. Our population has changed again."

"I see," Keisha adds, "That is $256+3\times 5-2+3$ that equals 1296 birds, I think. I'm not sure, that doesn't seem right."

We have an equation that Keisha wrote to represent the comings and goings of the birds in the aviary.

Before we figure out if Keisha's math is correct, let's underline any important information in the problem.

Wow, there is a lot going on. Here is what we have to work with.

256 birds

3×5 - three birds each gave birth to five baby birds

1. birds were released
2. injured birds were found.

Since we started with 256 birds, that begins our equation. Then we can add in all of the pieces of the problem.

$256+3\times 5-2+3=$ ___

This is the same equation that Keisha came up with. Let's look at her math.

Keisha says, "That is $256+3\times 5-2+3$ that equals 1296 birds, I think. I'm not sure, that doesn't seem right."

It isn't correct. Keisha forgot to use the order of operations.

According to the order of operations, Keisha needed to multiply 3×5 BEFORE completing any of the other operations.

Let's look at that.

$$\begin{aligned} 256+3\times 5-2+3 &= \_\_\_ \\ 256+15-2+3 &= \_\_\_ \\ 256+13+3 &= \_\_\_ \\ 269+3 &= \_\_\_ \\ 272 &= \_\_\_ \end{aligned}$$

Now we can complete the addition and subtraction in order from left to right.

$256+15-2+3=272$

The new bird count in the aviary is 272 birds.

Practice Set

Evaluate each expression according to the order of operations.

1. $2+3\times4+7=$ ___
2. $4+5\times2+9-1=$ ___
3. $6\times7+2\times3=$ ___
4. $4\times5+3\times1-9=$ ___
5. $5\times3\times2+5-1=$ ___
6. $4+7\times3+8\times2=$ ___
7. $9-3\times1+4-7=$ ___
8. $10+3\times4+2-8=$ ___
9. $11\times3+2\times4-3=$ ___
10. $6+7\times8-9\times2=$ ___
11. $3+4^2-5\times2+9=$ ___
12. $2^2+5\times2+6^2-11=$ ___
13. $3^2\times2+4-9=$ ___
14. $6+3\times2^2+7-1=$ ___
15. $7+2\times4+3^2-5=$ ___
16. $3+(2+7)-3+5=$ ___
17. $2+(5-3)+7^2-11=$ ___
18. $4\times2+(6-4)-9+5=$ ___
19. $8^2-4+(9-3)+12=$ ___
20. $7^3-100+(3+4)-9=$ ___

Check each answer using order of operations. Write whether the answer is true or false.

21. $4+5\times2+8-7=15$
22. $4+3\times9+6-10=104$
23. $6+2^2\times4+3\times6=150$
24. $3+6\times3+9\times7-18=66$
25. $7\times2^3+4-9\times3-8=25$

Directions: Insert grouping symbols to make each a true statement.

26. $4+5-2+3-2=8$

27. $2+3\times 2-4=6$

28. $1+9\times 4+3+2-1=121$

29. $7+4\times 3-5\times 2=23$

30. $2^2+5\times 8-3+4=33$

1.5 Chapter 1 Review

Compare the real numbers. Which real number is the largest?

1. 7 *and* -11
2. $\frac{4}{5}$ *and* $\frac{11}{16}$
3. $\frac{10}{15}$ *and* $\frac{2}{3}$
4. 0.985 *and* $\frac{31}{32}$
5. -16.12 *and* $\frac{-300}{9}$

Order the real numbers from least to greatest.

6. $\frac{8}{11}, \frac{7}{10}, \frac{5}{9}$

7. $\frac{2}{7}, \frac{1}{11}, \frac{8}{13}, \frac{4}{7}, \frac{8}{9}$

Graph these values on the same number line.

8. $3\frac{1}{3}$

9. -1.875

10. $\frac{7}{8}$

11. $0.1\bar{6}$

12. $\frac{-55}{5}$

Simplify by performing the operation(s).

13. $\frac{8}{5} - \frac{4}{3}$

14. $\frac{4}{3} - \frac{1}{2}$

15. $\frac{1}{6} + 1\frac{5}{6}$

16. $\frac{-5}{4} \times \frac{1}{3}$

17. $\frac{4}{9} \times \frac{7}{4}$

18. $-1\frac{5}{7} \times -2\frac{1}{2}$

19. $\frac{1}{9} \div -1\frac{1}{3}$

20. $\frac{-3}{2} \div \frac{-10}{7}$

21. $-3\frac{7}{10} \div 2\frac{1}{4}$

22. $1\frac{1}{5} - \left(-3\frac{3}{4}\right)$

23. $4\frac{2}{3} + 3\frac{2}{3}$

24. $5.4 + (-9.7)$

25. $(-7.1) + (-0.4)$

26. $(-4.79) + (-3.63)$

27. $(-8.1) - (-8.9)$

28. $1.58 - (-13.6)$

29. $(-13.6) + 12 - (-15.5)$

30. $(-5.6)-(-12.6)+(-6.6)$

31. $19.4+24.2$

32. $8.7+3.8+12.3$

33. $9.8-9.4$

34. $2.2-7.3$

Which property has been applied?

35. $6.78+(-6.78)=0$

36. $9.8+11.2+1.2=9.8+1.2+11.2$

37. $\frac{4}{3}-\left(-\frac{5}{6}\right)=\frac{4}{3}+\frac{5}{6}$

38. $8(11)\left(\frac{1}{8}\right)=8\left(\frac{1}{8}\right)(11)$

Solve the real-world situation.

39. Carol has 18 feet of fencing and purchased an addition 132 inches. How much fencing does Carol have?

40. Ulrich is making cookies for a fundraiser. Each cookie requires $\frac{3}{8}$ pound of dough. He has 12 pounds of cookie dough. How many cookies can Ulrich make?

41. Bagger 288 is a trench digger, which moves at $\frac{3}{8}$ $miles/hour$. How long will it take to dig a trench 14 miles long?

42. Georgia started with a given amount of money, a. She spent \$4.80 on a large latte, \$1.20 on an English muffin, \$68.48 on a new shirt, and \$32.45 for a present. She now has \$0.16. How much money, a, did Georgia have in the beginning?

43. The formula for an area of a square is $A=s^2$. A square garden has an area of 145 meters^2. Find the length of the garden exactly.

CHAPTER 2 Introduction to Variables

Chapter Outline

In order to represent real life situations mathematically, we often use symbols to represent unknown quantities. We call these symbols variables. Each mathematical subject requires knowledge of manipulating expressions and equations to solve for a variable. Careers such as automobile accident investigators, quality control engineers, and insurance originators use equations to determine the value of variables.

Throughout this chapter, you will learn how variables can be used to represent unknown quantities in various situations, and how to simplify algebraic expressions with variables using the Order of Operations.

2.1 Variable Expressions

Who Speaks Math, Anyway?

When someone is having trouble with algebra, they may say, “I don’t speak math!” While this may seem weird to you, it is a true statement. Math, like English, French, Spanish, or Arabic, is a like a language that you must learn in order to be successful. In order to understand math, you must practice the language.

Action words, like run, jump, or drive, are called verbs. In mathematics, **operations** are like verbs because they involve doing something. Some operations are familiar, such as addition, multiplication, subtraction, or division. Operations can also be much more complex like an raising to an exponent or taking a square root.

Example 1: Suppose you have a job earning $8.15 per hour. What could you do to quickly find out how much money you would earn for different hours of work?

Solution: You could make a list of all the possible hours, but that would take forever! So instead, you let the “hours you work” be replaced with a symbol, like h for hours, and write an equation such as:

$$amount\ of\ money = 8.15h$$

In mathematics, variable is a symbol, usually an English letter, written to replace an unknown or changing quantity.

Example 2: What variable symbol would be a good choice for the following situations?

a. the number of cars on a road

b. time in minutes of a ball bounce

c. distance from an object

Solution: There are many options, but here are a few to think about.

a. Cars is the changing value, so c is a good choice.

b. Time is the changing value, so t is a good choice.

c. Distance is the varying quantity, so d is a good choice.

Why Do They Do That?

Just like in the English language, mathematics uses several words to describe one thing. For example, *sum, addition, more than,* and *plus* all mean to add numbers together. The following definition shows an example of this.

Definition: To **evaluate** means to complete the operations in the math sentence. **Evaluate** can also be called simplify or answer.

To begin to evaluate a mathematical **expression**, you must first **substitute** a number for the variable.

Definition: To **substitute** means to replace the variable in the sentence with a value.

Now try out your new vocabulary.

Example 3: EVALUATE $7y-11$, when $y=4$.

Solution: Evaluate means to follow the directions, which is to take 7 times y and subtract 11. Because y is the number 4,

$7\times4-11$	We have "substituted" the number 4 for y.
$28-11$	Multiplying 7 and 4
17	Subtracting 11 from 28
The solution is 17.	

Because algebra uses variables to represent the unknown quantities, the multiplication symbol $\times$ is often confused with the variable x. To help avoid confusion, mathematicians replace the multiplication symbol with parentheses ( ), the multiplication dot $\cdot$, or by writing the expressions side by side.

Example 4: Rewrite $P=2\times l+2\times w$ with alternative multiplication symbols.

Solution: $P=2\times l+2\times w$ can be written as $P=2\cdot l+2\cdot w$

It can also be written as $P=2l+2w$.

The following is a real-life example that shows the importance of evaluating a mathematical variable.

Example 5: To prevent major accidents or injuries, horses must be fenced in a rectangular pasture. If the dimensions of the pasture are 300 feet by 225 feet, how much fencing should the ranch hand purchase to enclose the pasture?

Solution: Begin by drawing a diagram of the pasture and labeling what you know.

To find the amount of fencing needed, you must add all the sides together;

$$L+L+W+W.$$

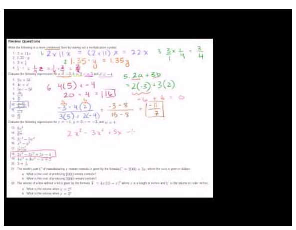
By substituting the dimensions of the pasture for the variables L and W, the expression becomes

$$300+300+225+225.$$

Now we must evaluate by adding the values together. The ranch hand must purchase 1,050 feet of fencing.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. CK-12 Basic Algebra:Variable Expressions (12:26)



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In 1 – 5, evaluate the expression.

1. $5m+7$ *when* $m=3$.
2. $\frac{1}{3}(c)$ *when* $c=63$.
3. $\$8.15(h)$ *when* $h=40$.
4. $(k-11)\div 8$ *when* $k=43$.
5. Evaluate $(-2)^2+3(j)$ *when* $j=-3$.

In 6 – 13, evaluate the expressions. Let $a=-3,\ b=2,\ c=5,$ *and* $d=-4$.

6. $2a+3b$

7. $4c+d$

8. $5ac - 2b$

9. $\frac{2a}{c-d}$

10. $\frac{3b}{d}$

11. $\frac{a-4b}{3c+2d}$

12. $\frac{1}{a+b}$

13. $\frac{ab}{cd}$

In 14 – 21, evaluate the expressions. Let $x = -1,\ y = 2,\ z = -3$, and $w = 4$.

14. $8x^3$

15. $\frac{5x^2}{6z^3}$

16. $3z^2 - 5w^2$

17. $x^2 - y^2$

18. $\frac{z^3+w^3}{z^3-w^3}$

19. $2x^2 - 3x^2 + 5x - 4$

20. $4w^3 + 3w^2 - w + 2$

21. $3 + \frac{1}{z^2}$

In 22 – 26, choose an appropriate variable to describe each situation.

22. The number of hours you work in a week

23. The distance you travel

24. The height of an object over time

26. The area of a square

27. The number of steps you take in a minute In 27 – 31, evaluate the real-life problems.

28. The measurement around the widest part of these holiday bulbs is called their *circumference.* The formula for circumference is $2\pi r$, where $\pi \approx 3.14$ and r is the radius of the circle. Suppose the radius is 1.25 inches. Find the *circumference.*

FIGURE 2.1
Christmas Baubles by Petr Kratochvil

29. The dimensions of a piece of notebook paper are 8.5 inches by 11 inches. Evaluate the writing area of the paper. The formula for area is length $\times$ width.

30. Sonya purchases 16 cans of soda at \$0.99 each. What is the amount Sonya spent on soda?

31. Mia works at a job earning \$4.75 per hour. How many hours should she work to earn \$124.00?

32. The area of a square is the side length squared. Evaluate the area of a square with side length 10.5 miles.

2.2 Patterns and Expressions

In mathematics, especially in algebra, we look for patterns in the numbers that we see. Using mathematical verbs and variables studied in previous lessons, expressions can be written to describe a pattern.

Definition: An **algebraic expression** is a mathematical phrase combining numbers and/or variables using mathematical operations.

Consider a theme park charging admission of $28 per person. In this case, the total amount of money collected can be described by the phrase, "*Twenty-eight times the number of people who enter the park.*"

The English phrase above can be translated (to write in another language) into an algebraic expression.

Example 1: Write an expression to describe the amount of revenue of the theme park.

Solution: An appropriate variable to describe the number of people could be p. Rewriting the English phrase into a mathematical phrase, it becomes $28 \times p$.

Example 2: Write an algebraic expression for the following phrase.

The product of c and 4.

Solution: The verb is *product*, meaning "to multiply." Therefore, the phrase is asking for the answer found by multiplying c and 4. The nouns are the number 4 and the variable c. The expression becomes $4 \times c$, $4(c)$, or using shorthand, $4c$.

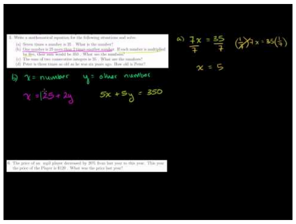
Example 3: Write an algebraic expression for the following phrase.

3 times the sum of c and 4.

Solution: In this example, the phrase consists of "3 times" followed by the phrase "the sum of c and 4". If we put the second phrase in parenthesis, we get $3 \times$ (the sum of c and 4). This can be shortened to $3(c+4)$.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. CK-12 Basic Algebra:Patterns and Equations (13:18)



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For exercises 1 – 15, translate the English phrase into an algebraic expression. For the exercises without a stated variable, choose a letter to represent the unknown quantity.

1. Sixteen more than a number
2. The quotient of h and 8
3. Forty-two less than y
4. The product of k and three
5. The sum of g and -7
6. r minus 5.8
7. 6 more than 5 times a number
8. 6 divided by a number minus 12
9. A number divided by -11
10. 27 less than a number times four
11. The quotient of 9.6 and m
12. 2 less than 10 times a number
13. The quotient of d and five times s
14. 35 less than x
15. The product of 6, -9, and u

In exercises 16 – 24, write an English phrase for each algebraic expression

16. $J-9$
17. $\frac{n}{14}$
18. $17-a$
19. $3l-16$
20. $\frac{1}{2}(h)(b)$
21. $\frac{b}{3}+\frac{z}{2}$
22. $4.7-2f$
23. $5.8+k$
24. $2l+2w$

In exercises 25 – 28, define a variable to represent the unknown quantity and write an expression to describe the situation.

25. The *unit cost* represents the quotient of the total cost and number of items purchased. Write an expression to represent the unit cost of the following: The total cost is \$14.50 for n objects.
26. The area of a square is the side length squared.
27. The total length of ribbon needed to make dance outfits is 15 times the number of outfits.
28. What is the remaining amount of chocolate squares if you started with 16 and have eaten some?

29. Describe a real-world situation that can be represented by $h+9$.
30. What is the difference between $\frac{7}{m}$ and $\frac{m}{7}$?

2.3 Combining Like Terms

In the last lesson you learned how to write single-variable expressions and single variable equations. Now you are going to learn to work with single-variable expressions. The first thing that you are going to learn is how to ***simplify*** an expression.

What does it mean to simplify?

To simplify means to make smaller or to make simpler. When we simplify in mathematics, we aren't solving anything, we are just making it smaller.

How do we simplify expressions?

Sometimes, you will be given an expression using variables where there is more than one term. **A term** can be either a number or or a variable or it can be a number multiplied by a variable. As an example, take a look at the following expression:

$4x+3$

This expression consists of two terms. The first term is $4x$. It is a number and a variable. The second term is the 3. It does not have a variable in it, but it is still a term.

We haven't been given a value for x, so there isn't anything else we can do with this expression. It stays the same. If we have been given a value for x, then we could evaluate the expression. You have already worked on evaluating expressions.

When there is more than one LIKE TERM in an expression, we can simplify the expression.

What is a like term?

A like term means that the terms in question use the same variable, raised to the same power.

$4x$ **and** $5x$ **are like terms. They both have** x **as the variable. They are alike.**

$6x$ **and** $2y$ **are not like terms. One has an** x **and one has a** y**. They are not alike.**

When expressions have like terms, they can be simplified. We can simplify the sums and differences of expressions with like terms. Let's start with sums. Here is an example:

Example 1

$$5x+7x$$

First, we look to see if these terms are alike. Both of them have an x **so they are alike.**

Next, we can simplify them by adding the numerical part of the terms together. The x **stays the same.**

$$\begin{aligned}&5x+7x\\&=12x\end{aligned}$$

You can think of the x as a label that lets you know that the terms are alike.

Let's look at another example.

Example 2

$$7x + 2x + 5y$$

First, we look to see if the terms are alike. Two of the terms have x's and one has a y. The two with the x's are alike. The one with the y is not alike. We can simplify the ones with the x's.

Next, we simplify the like terms.

$$7x + 2x = 9x$$

We can't simplify the $5y$ so it stays the same.

$$9x + 5y$$

This is our answer.

We can also simplify expressions with differences and like terms. Let's look at an example.

Example 3

$$9y - 2y$$

First, you can see that these terms are alike because they both have y's. We simplify the expression by subtracting the numerical part of the terms.

9 - 2 = 7

Our answer is $7y$.

Sometimes you can combine like terms that have both sums and differences in the same example.

Example 4

$$8x - 3x + 2y + 4y$$

We begin with the like terms.

$$8x - 3x = 5x$$
$$2y + 4y = 6y$$

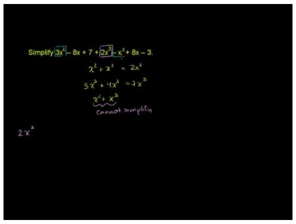
Next, we put it all together.

$$5x + 6y$$

This is our answer.

Remember that you can only combine terms that are alike!!!

For further explanation, watch the video on simplifying expressions and combininglike terms:



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Practice Set

Simplify the expressions by combining like terms.

1. $7z+2z+4z$
2. $25y-13y$
3. $7x+2x+4a$
4. $45y-15y+13y$
5. $-32m+12m$
6. $-6x+7x-12x$
7. $14a+18b-5a-8b$
8. $-32m+12m$
9. $-11t-12t-7t$
10. $15w+7h-15w+21h$
11. $-7x+39x$
12. $3x^2+21x+5x+10x^2$
13. $6xy+7y+5x+9xy$
14. $10ab+9-2ab$
15. $-7mn-2mn^2-2mn+8$

2.4 The Distributive Property

When we multiply an algebraic expression by a number or by another algebraic expression, we need to use the distributive property.

The **Distributive Property:** For any real numbers or expressions ***A, B*** and **C**:

A(B+C) = AB + AC

A(B-C) = AB - AC

Example 1: *Determine the value of* $11(2+6)$ *using both order of operations and the Distributive Property.*

Solution: Using the order of operations: $11(2+6) = 11(8) = 88$

Using the Distributive Property: $11(2+6) = 11(2)+11(6) = 22+66 = 88$

Regardless of the method, the answer is the same.

Example 2: *Simplify* $7(3x-5)$

Solution 1: Think of this expression as seven groups of $(3x-5)$. You could write this expression seven times and add all the like terms. $(3x-5)+(3x-5)+(3x-5)+(3x-5)+(3x-5)+(3x-5)+(3x-5) = 21x-35$

Solution 2: Apply the Distributive Property. $7(3x-5) = 7(3x)+7(-5) = 21x-35$

Example 3: *Simplify* $\frac{2}{7}(3y^2-11)$

Solution: Apply the Distributive Property.

$$\frac{2}{7}(3y^2+-11) = \frac{2}{7}(3y^2)+\frac{2}{7}(-11)$$
$$\frac{6y^2}{7}-\frac{22}{7}$$

Identifying Expressions Involving the Distributive Property

The Distributive Property often appears in expressions, and many times it does not involve parentheses as grouping symbols. Previously, we saw how the fraction bar acts as a grouping symbol. The following example involves using the Distributive Property with fractions.

Example 4: *Simplify* $\frac{2x+4}{8}$

Solution: Think of the denominator as: $\frac{2x+4}{8} = \frac{1}{8}(2x+4)$

Now apply the Distributive Property: $\frac{1}{8}(2x) + \frac{1}{8}(4) = \frac{2x}{8} + \frac{4}{8}$

Simplify: $\frac{x}{4} + \frac{1}{2}$

Solve Real-World Problems Using the Distributive Property

The Distributive Property is one of the most common mathematical properties seen in everyday life. It crops up in business and in geometry. Anytime we have two or more groups of objects, the Distributive Property can help us solve for an unknown.

Example 5: *An octagonal gazebo is to be built as shown below. Building code requires five-foot long steel supports to be added along the base and four-foot long steel supports to be added to the roof-line of the gazebo. What length of steel will be required to complete the project?*

Solution: Each side will require two lengths, one of five and four feet respectively. There are eight sides, so here is our equation.

Steel required $= 8(4+5)$ feet.

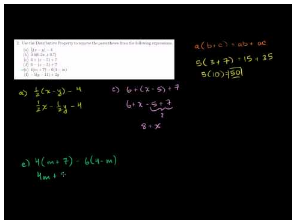
We can use the distributive property to find the total amount of steel:

Steel required $= 8 \times 4 + 8 \times 5 = 32 + 40$ feet.

A total of 72 feet of steel is needed for this project.

Practice Set: Distributive Property

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. CK-12 Basic Algebra:Distributive Property (5:39)



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Use the Distributive Property to simplify the following expressions.

1. $(x+4)-2(x+5)$
2. $\frac{1}{2}(4z+6)$
3. $(4+5)-(5+2)$
4. $(x+2+7)$
5. $0.25(6q+32)$
6. $y(x+7)$
7. $-4.2(h-11)$
8. $13x(3y+z)$
9. $\frac{1}{2}(x-y)-4$
10. $0.6(0.2x+0.7)$
11. $(2-j)(-6)$
12. $(r+3)(-5)$
13. $6+(x-5)+7$
14. $6-(x-5)+7$
15. $4(m+7)-6(4-m)$
16. $-5(y-11)+2y$

Use the Distributive Property to simplify the following fractions.

17. $\frac{8x+12}{4}$
18. $\frac{9x+12}{3}$
19. $\frac{11x+12}{2}$
20. $\frac{3y+2}{6}$
21. $-\frac{6z-2}{3}$
22. $\frac{7-6p}{3}$

In 23 – 25, write an expression for each phrase.

23. $\frac{2}{3}$ times the quantity of n plus 16
24. Twice the quantity of m minus 3
25. $-4x$ times the quantity of x plus 2
26. A bookshelf has five shelves, and each shelf contains seven poetry books and eleven novels. How many of each type of book does the bookcase contain?
27. Use the Distributive Property to show how to simplify 6(19.99) in your head.
28. A student rewrote $4(9x+10)$ as $36x+10$. Explain the student's error.
29. Use the Distributive Property to simplify 9(5998) in your head.

30. Amar is making giant holiday cookies for his friends at school. He makes each cookie with 6 oz of cookie dough and decorates them with macadamia nuts. If Amar has 5 lbs of cookie dough $(1\ lb = 16\ oz)$ and 60 macadamia nuts, calculate the following.
    (a) How many **(full)** cookies can he make?
    (b) How many macadamia nuts can he put on each cookie, if each is supposed to be identical?

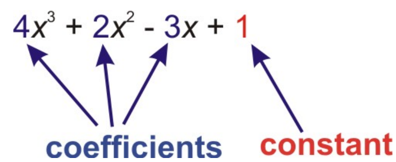
2.5 Addition and Subtraction of Polynomials

Some algebraic expressions are called polynomials. A **polynomial** is made up of different terms that contain **positive integer** powers of the variables. Here is an example of a polynomial:

$$4x^3+2x^2-3x+1$$

The example above is a polynomial with *four terms*.

The numbers appearing in each term in front of the variable are called the **coefficients**. The number appearing all by itself without a variable is called a **constant term**.



In this case the coefficient of x^3 is **4**, the coefficient of x^2 is **2**, the coefficient of x is **-3** and the constant term is **1**.

Degrees of Polynomials and Standard Form

Each term in the polynomial has a different **degree**. The degree of the term is the power of the variable in that term. A constant term is said to have a degree of 0.

$4x^3$	has degree 3 or 3^{rd} order term.
$2x^2$	has degree 2 or 2^{nd} order term.
$-3x$	has degree 1 or 1^{st} order term.
1	has degree 0 and is called the constant.

By definition, **the degree of the polynomial** is the same as the term with the highest degree. This example is a polynomial of degree 3.

Example 1

For the following polynomials, identify the coefficient of each term, the constant, the degree of each term and the degree of the polynomial.

a) $x^5-3x^3+4x^2-5x+7$

Solution

a) $x^5-3x^3+4x^2-5x+7$

The coefficients of each term in order are 1, -3, 4, and -5 and the constant is 7.

The degrees of each term are 5, 3, 2, 1, and 0. Therefore the degree of the polynomial is 5.

Often, we arrange the terms in a polynomial in order of decreasing power. This is called **standard form**.

The following polynomials are in standard form:

$$4x^4 - 3x^3 + 2x^2 - x + 1$$

$$a^4 - 2a^3 + 3a^2 - 5a + 2$$

The first term of a polynomial in standard form is called the **leading term**, and the coefficient of the leading term is called the **leading coefficient**.

The first polynomial above has the leading term $4x^4$, and the leading coefficient is 4.

The second polynomial above has the leading term a^4, and the leading coefficient is 1.

Example 2

Rearrange the terms in the following polynomials so that they are in standard form. Indicate the leading term and leading coefficient of each polynomial.

a) $7 - 3x^3 + 4x$

b) $a - a^3 + 2$

c) $-4b + 4 + b^2$

Solution

a) $7 - 3x^3 + 4x$ becomes $-3x^3 + 4x + 7$. Leading term is $-3x^3$; leading coefficient is -3.

b) $a - a^3 + 2$ becomes $-a^3 + a + 2$. Leading term is $-a^3$; leading coefficient is -1.

c) $-4b + 4 + b^2$ becomes $b^2 - 4b + 4$. Leading term is b^2; leading coefficient is 1.

Simplifying Polynomials

A polynomial is simplified if it has no terms that are alike. Recall that **Like terms** are terms in the polynomial that have the same variable(s) with the same exponents, whether they have the same or different coefficients.

When a polynomial has like terms, we can simplify it by combining those terms.

$$x^2 + \underline{6x} - \underline{4x} + 2$$

Like terms

We can simplify this polynomial by combining the like terms $6x$ and $-4x$ into $(6-4)x$, or $2x$. The new polynomial is $x^2 + 2x + 2$.

Example 3

Simplify the following polynomial by collecting like terms and combining them.

$2x - 4x^2 + 6 + x^2 - 4 + 4x$

Solution

Rearrange the terms so that like terms are grouped together: $(-4x^2 + x^2) + (2x + 4x) + (6 - 4)$

Combine each set of like terms: $-3x^2 + 6x + 2$

Simplifying Polynomials using the Distributive Property

We need to employ the Distributive Property to simplify expressions that involve multiplying a polynomial by a leading coefficient. This is also often required when we add or subtract polynomial expressions. In the last section we learned the distributive property:

Distributive Property: ***A(B + C) = AB + AC***

When multiplying a polynomial by a leading coefficient, we multiply the coefficient by each term of the polynomial. For example, let's multiply the polynomial 2x + 6 by 4. First we write this expression as 4(2x + 6). Then we distribute by multiplying each term of the polynomial by 4.

4(5x + 6) = 4(5x) + 4(6) = 20x + 24

Example 4

Distributing a leading coefficient through a polynomial

a) Simplify 3(2x+6)

b) Simplify -4(3x - 2)

c) Simplify 2 + 4(6x - 4)

d) Simplify 3 -(2x - 3)

Solution

a) Distribute the 3 through the polynomial by multiplying it by each term

$3(2x+6) = 3(2x)+3(6) = 6x+18$

b) Distribute the -4 through the polynomial by multiplying it by each term

$-4(2x+6) = (-4)(2x)+(-4)(6) = -8x-24$

c) In this example, we need to make sure we distribute the 4 through the polynomial before we add the 2. Then we combine like terms for our final answer.

$2+4(6x-4) = 2+4(6x)+4(-4) = 2+24x-16 = 24x-14$

d) In the last example, we are presented with a number minus a polynomial. In this case, we have to distribute the negative sign through the polynomial and then combine like terms. To do this, we convert the expression to an addition problem.

$$\begin{aligned}3-(2x-4) &= 3+(-1)(2x-4)\\ \text{Distribute:} &= 3+(-1)(2x)-(-1)(4)\\ \text{Combine Like Terms:} &= 3-2x+4\\ &= -2x+7\end{aligned}$$

Simplifying when Adding and Subtracting Polynomials

To add two or more polynomials, write their sum and then simplify by combining like terms.

Example 6a

Add and simplify the resulting polynomial.

Add $3x^2-4x+7$ and $2x^3-4x^2-6x+5$

Solution

$$
\begin{aligned}
& (3x^2-4x+7)+(2x^3-4x^2-6x+5) \\
\text{Group like terms: } & = 2x^3+(3x^2-4x^2)+(-4x-6x)+(7+5) \\
\text{Simplify: } & = 2x^3-x^2-10x+12
\end{aligned}
$$

To subtract one polynomial from another, distribute a -1 to each term of the polynomial you are subtracting.

Example 7

a) Subtract $x^3-3x^2+8x+12$ from $4x^2+5x-9$

b) Subtract $5b^2-2a^2$ from $4a^2-8ab-9b^2$

Solution

a)

$$
\begin{aligned}
(4x^2+5x-9)-1(x^3-3x^2+8x+12) & = 4x^2+5x-9-x^3+3x^2-8x-12 \\
\text{Group like terms: } & = -x^3+(4x^2+3x^2)+(5x-8x)+(-9-12) \\
\text{Simplify: } & = -x^3+7x^2-3x-21
\end{aligned}
$$

b)

$$
\begin{aligned}
(4a^2-8ab-9b^2)-1(5b^2-2a^2) & = 4a^2-8ab-9b^2-5b^2+2a^2 \\
\text{Group like terms: } & = (4a^2+2a^2)+(-9b^2-5b^2)-8ab \\
\text{Simplify: } & = 6a^2-14b^2-8ab
\end{aligned}
$$

Note: An easy way to check your work after adding or subtracting polynomials is to substitute a convenient value in for the variable, and check that your answer and the problem both give the same value. For example, in part (b) above, if we let $a=2$ and $b=3$, then we can check as follows:

Given	Solution
$(4a^2-8ab-9b^2)-(5b^2-2a^2)$	$6a^2-14b^2-8ab$
$(4(2)^2-8(2)(3)-9(3)^2)-(5(3)^2-2(2)^2)$	$6(2)^2-14(3)^2-8(2)(3)$
$(4(4)-8(2)(3)-9(9))-(5(9)-2(4))$	$6(4)-14(9)-8(2)(3)$
$(-113)-37$	$24-126-48$
-150	-150

Since both expressions evaluate to the same number when we substitute in arbitrary values for the variables, we can be reasonably sure that our answer is correct.

Note: When you use this method, do not choose 0 or 1 for checking since these can lead to common problems.

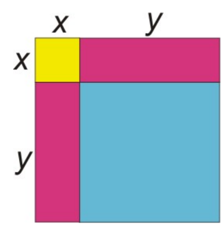
Problem Solving Using Addition or Subtraction of Polynomials

One application that uses polynomials is finding the area of a geometric figure.

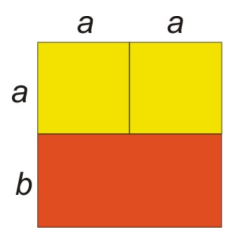
Example 7

Write a polynomial that represents the area of each figure shown.

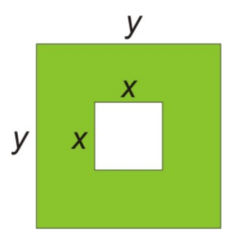
a)



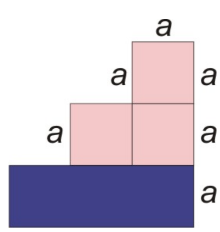
b)



c)



d)



Solution

a) This shape is formed by two squares and two rectangles.

$$\text{The blue square has area } y \times y = y^2.$$
$$\text{The yellow square has area } x \times x = x^2.$$
$$\text{The pink rectangles each have area } x \times y = xy.$$

To find the total area of the figure we add all the separate areas:

$$\begin{aligned} Total\ area &= y^2 + x^2 + xy + xy \\ &= y^2 + x^2 + 2xy \end{aligned}$$

b) This shape is formed by two squares and one rectangle.

$$\text{The yellow squares each have area } a \times a = a^2.$$
$$\text{The orange rectangle has area } 2a \times b = 2ab.$$

To find the total area of the figure we add all the separate areas:

$$\begin{aligned} Total\ area &= a^2 + a^2 + 2ab \\ &= 2a^2 + 2ab \end{aligned}$$

c) To find the area of the green region we find the area of the big square and subtract the area of the little square.

$$\begin{aligned} \text{The big square has area} : y \times y &= y^2. \\ \text{The little square has area} : x \times x &= x^2. \\ Area\ of\ the\ green\ region &= y^2 - x^2 \end{aligned}$$

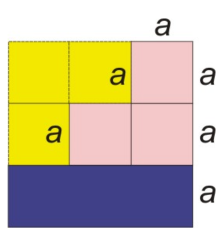
d) To find the area of the figure we can find the area of the big rectangle and add the areas of the pink squares.

$$\text{The pink squares each have area } a \times a = a^2.$$
$$\text{The blue rectangle has area } 3a \times a = 3a^2.$$

To find the total area of the figure we add all the separate areas:

$$Total\ area = a^2 + a^2 + a^2 + 3a^2 = 6a^2$$

Another way to find this area is to find the area of the big square and subtract the areas of the three yellow squares:



$$\text{The big square has area } 3a \times 3a = 9a^2.$$
$$\text{The yellow squares each have area } a \times a = a^2.$$

To find the total area of the figure we subtract:

$$\begin{aligned} Area &= 9a^2 - (a^2 + a^2 + a^2) \\ &= 9a^2 - 3a^2 \\ &= 6a^2 \end{aligned}$$

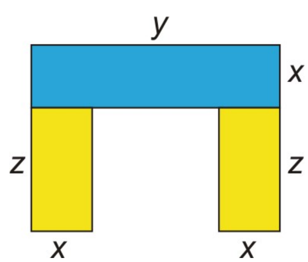
Further Practice

For more practice adding and subtracting polynomials, try playing the Battleship game at http://www.quia.com/ba/28820.html. (The problems get harder as you play; watch out for trick questions!)

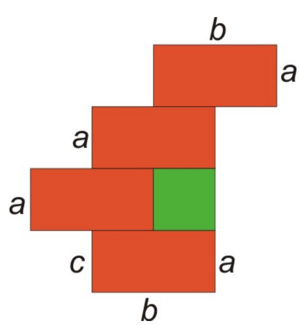
Practice Set

Express each polynomial in standard form. Give the degree of each polynomial.

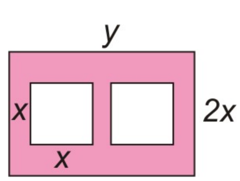
1. $3-2x$
2. $8-4x+3x^3$
3. $-5+2x-5x^2+8x^3$
4. x^2-9x^4+12
5. $5x+2x^2-3x$ Add and simplify.
6. $(x+8)+(-3x-5)$
7. $(-2x^2+4x-12)+(7x+x^2)$
8. $(2a^2b-2a+9)+(5a^2b-4b+5)$
9. $(6.9a^2-2.3b^2+2ab)+(3.1a-2.5b^2+b)$
10. $\left(\frac{3}{5}x^2-\frac{1}{4}x+4\right)+\left(\frac{1}{10}x^2+\frac{1}{2}x-2\frac{1}{5}\right)$ Subtract and simplify.
11. $(-t+5t^2)-(5t^2+2t-9)$
12. $(-y^2+4y-5)-(5y^2+2y+7)$
13. $(-5m^2-m)-(3m^2+4m-5)$
14. $(2a^2b-3ab^2+5a^2b^2)-(2a^2b^2+4a^2b-5b^2)$
15. $(3.5x^2y-6xy+4x)-(1.2x^2y-xy+2y-3)$
16. Find the area of the following figures.



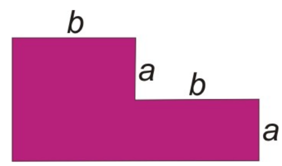
17.



18.



19.



CHAPTER 3 Polynomials and Exponents, Part 1

Chapter Outline

As we saw in the last section, polynomial expressions often involve variables with **exponents**. We learned how to add and subtract and simplify these expressions in the last chapter. This chapter focuses on multiplying expressions. We will learn the properties of exponents which will will be important to simplify our expressions, and we will learn how to multiply polynomials.

Definition: An **exponent** is a power of a number which shows how many times that number is multiplied by itself.

An example would be 2^3. You would multiply 2 by itself 3 times, $2 \times 2 \times 2$. The number 2 is the **base** and the number 3 is the **exponent.** The value 2^3 is called the **power.**

Example 1: *Write in exponential form:* $a \cdot a \cdot a \cdot a$

Solution: You must count the number of times the base, a is being multiplied by itself. It's being multiplied 4 times so the solution is a^4

Note: When you raise negative numbers to a power, you need to keep track of the negatives. Recall that

$$(negative\ number) \times (positive\ number) = negative\ number$$
$$(negative\ number) \times (negative\ number) = positive\ number$$

For **even powers** of negative numbers, the answer will always be positive. Pairs can be made with each number and the negatives will be cancelled out.

$$(-2)^4 = (-2)(-2)(-2)(-2) = (-2)(-2) \cdot (-2)(-2) = +16$$

For **odd powers** of negative numbers, the answer is always negative. Paris can be made but there will still be one negative number unpaired making the answer negative.

$$(-2)^5 = (-2)(-2)(-2)(-2)(-2) = (-2)(-2) \cdot (-2)(-2) \cdot (-2) = -32$$

3.1 Exponential Properties Involving Products

When we multiply the same numbers, each with different powers, it is easier to combine them before solving. This is why we use the **Product of Powers Property.**

Product of Powers Property: for all real numbers $x, x^n \cdot x^m = x^{n+m}$

Example 1: *Multiply* $x^4 \cdot x^5$

Solution: We could write this in expanded form to see that:

$$(x^4)(x^5) = \underbrace{(x \cdot x \cdot x \cdot x)}_{x^4} \cdot \underbrace{(x \cdot x \cdot x \cdot x \cdot x)}_{x^5} = \underbrace{(x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x)}_{x^9}$$

Or we could use the product of powers property to find the same result:

$x^4 \cdot x^5 = x^{4+5} = x^9$

Now let us consider the situation where we are raising an exponent to a power:

$$(x^4)^3 = x^4 \cdot x^4 \cdot x^4 \qquad \text{3 factors of } x \text{ to the power 4.}$$
$$\underbrace{(x \cdot x \cdot x \cdot x)}_{x^4} \cdot \underbrace{(x \cdot x \cdot x \cdot x)}_{x^4} \cdot \underbrace{(x \cdot x \cdot x \cdot x)}_{x^4} = \underbrace{(x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x)}_{x^{12}}$$

This situation is summarized below.

Power of a Product Property: for all real numbers x,

$$(x^n)^m = x^{n \cdot m}$$

The Power of a Product Property is similar to the Distributive Property. Everything inside the parentheses must be taken to the power outside. For example, $(x^2y)^4 = (x^2)^4 \cdot (y)^4 = x^8y^4$. Watch how it works the long way.

$$\underbrace{(x \cdot x \cdot y)}_{x^2y} \cdot \underbrace{(x \cdot x \cdot y)}_{x^2y} \cdot \underbrace{(x \cdot x \cdot y)}_{x^2y} \cdot \underbrace{(x \cdot x \cdot y)}_{x^2y} = \underbrace{(x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y)}_{x^8y4}$$

The Power of a Product Property does not work if you have a sum or difference inside the parenthesis. For example, $(x+y)^2 \neq x^2+y^2$. Because it is an addition equation, it should look like $(x+y)(x+y)$.

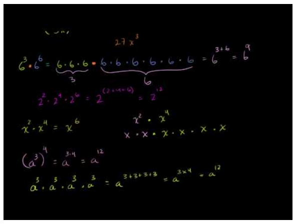
Example 2: *Simplify* $(x^2)^3$

Solution: $(x^2)^3 = x^{2 \cdot 3} = x^6$

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the

practice exercise listed in the following exercise set. However, the practice exercise is the same in both. Exponent Properties Involving Products (14:00)



MEDIA

Click image to the left for more content.

1. Consider a^5.

a. What is the base?

b. What is the exponent?

c. What is the power?

d. How can this power be written using repeated multiplication?

Determine whether the answer will be positive or negative. You do **not** have to provide the answer.

2. $-(3^4)$
3. -8^2
4. $10 \times (-4)^3$
5. What is the difference between -5^2 *and* $(-5)^2$? Enter both of these expressions, exactly as they appear, in your calculator. Why does the calculator give you different answers for each expression?

Write in exponential notation.

6. $2 \cdot 2$
7. $(-3)(-3)(-3)$
8. $y \cdot y \cdot y \cdot y \cdot y$
9. $(3a)(3a)(3a)(3a)$
10. $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$
11. $3x \cdot 3x \cdot 3x$
12. $(-2a)(-2a)(-2a)(-2a)$
13. $6 \cdot 6 \cdot 6 \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y$

Find each number:

14. 1^{10}
15. 0^3
16. 7^3
17. -6^2
18. 5^4

19. $3^4 \cdot 3^7$
20. $2^6 \cdot 2$
21. $(4^2)^3$
22. $(-2)^6$
23. $(0.1)^5$
24. $(-0.6)^3$

Multiply and simplify.

25. $6^3 \cdot 6^6$
26. $2^2 \cdot 2^4 \cdot 2^6$
27. $3^2 \cdot 4^3$
28. $x^2 \cdot x^4$
29. $x^2 \cdot x^7$
30. $(y^3)^5$
31. $(-2y^4)(-3y)$
32. $(4a^2)(-3a)(-5a^4)$

Simplify.

33. $(a^3)^4$
34. $(xy)^2$
35. $(3a^2b^3)^4$
36. $(-2xy^4z^2)^5$
37. $(3x^2y^3) \cdot (4xy^2)$
38. $(4xyz) \cdot (x^2y^3) \cdot (2yz^4)$
39. $(2a^3b^3)^2$
40. $(-8x)^3(5x)^2$
41. $(4a^2)(-2a^3)^4$
42. $(12xy)(12xy)^2$
43. $(2xy^2)(-x^2y)^2(3x^2y^2)$

3.2 Multiplying Two Polynomials

Let's start by multiplying two binomials together. A binomial is a polynomial with two terms, so a product of two binomials will take the form $(a+b)(c+d)$.

We can still use the Distributive Property here if we do it cleverly. First, let's think of the first set of parentheses as one term. The Distributive Property says that we can multiply that term by c, multiply it by d, and then add those two products together: $(a+b)(c+d) = (a+b)\cdot c + (a+b)\cdot d$.

We can rewrite this expression as $c(a+b)+d(a+b)$. Now let's look at each half separately. We can apply the distributive property again to each set of parentheses in turn, and that gives us $c(a+b)+d(a+b) = ca+cb+da+db$.

What you should notice is that when multiplying any two polynomials, *every term in one polynomial is multiplied by every term in the other polynomial.*

Example 1

Multiply and simplify: $(2x+1)(x+3)$

Solution

We must multiply each term in the first polynomial by each term in the second polynomial. Let's try to be systematic to make sure that we get all the products.

First, multiply the first term in the first set of parentheses by all the terms in the second set of parentheses.

FIGURE 3.1

$(2x+1)(x+3) = (2x)(x) + (2x)(3) + \ldots$

Now we're done with the first term. Next we multiply the second term in the first parenthesis by all terms in the second parenthesis and add them to the previous terms.

FIGURE 3.2

$(2x+1)(x+3) = (2x)(x) + (2x)(3) + (1)(x) + (1)(3)$

Now we're done with the multiplication and we can simplify:

$(2x)(x)+(2x)(3)+(1)(x)+(1)(3) = 2x^2+6x+x+3 = 2x^2+7x+3$

Example 2

Multiply and simplify:

a) $(4x-5)(x-20)$

b) $(3x-2)(3x+2)$

c) $(3x^2+2x-5)(2x-3)$

Solution

a) This would be

$$(4x-5)(x-20) = (4x)(x)+(4x)(-20)+(-5)(x)+(-5)(-20) = 4x^2-80x-5x+100 = 4x^2-85x+100$$

b)

$$(3x-2)(3x+2) = (3x)(3x)+(3x)(2)+(-2)(3x)+(-2)(2) = 9x^2+6x-6x-4 = 9x^2-4$$

c)

$$(3x^2+2x-5)(2x-3) = (3x^2)(2x)+(3x^2)(-3)+(2x)(2x)+(2x)(-3)+(-5)(2x)+(-5)(-3) = 6x^3-9x^2+4x^2-6x-10x+$$

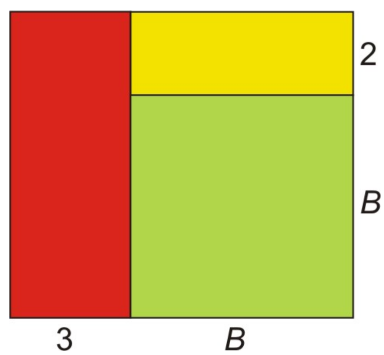
Solve Real-World Problems Using Multiplication of Polynomials

In this section, we'll see how multiplication of polynomials is applied to finding the areas and volumes of geometric shapes.

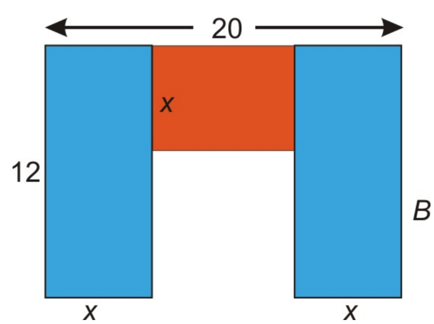
Example 3

Find the areas of the following figures:

a)

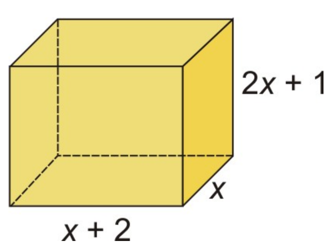


b)

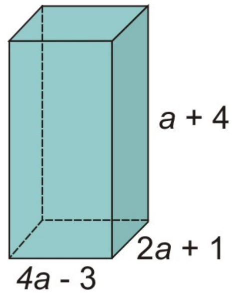


Find the volumes of the following figures:

c)



d)



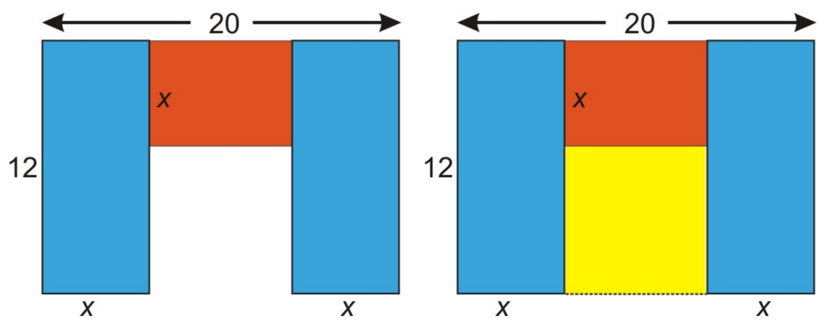
Solution

a) We use the formula for the area of a rectangle: $Area = \text{length} \times \text{width}$.

For the big rectangle:

$$\begin{aligned} Length &= b+3, \text{ Width} = b+2 \\ \text{Area} &= (b+3)(b+2) \\ &= b^2+2b+3b+6 \\ &= b^2+5b+6 \end{aligned}$$

b) We could add up the areas of the blue and orange rectangles, but it's easier to just find the area of the whole *big* rectangle and subtract the area of the yellow rectangle.



$$\begin{aligned} Areaofbigrectangle &= 20(12) = 240 \\ \text{Area of yellow rectangle} &= (12-x)(20-2x) \\ &= 240-24x-20x+2x^2 \\ &= 240-44x+2x^2 \\ &= 2x^2-44x+240 \end{aligned}$$

The desired area is the difference between the two:

$$\begin{aligned} Area &= 240-(2x^2-44x+240) \\ &= 240+(-2x^2+44x-240) \\ &= 240-2x^2+44x-240 \\ &= -2x^2+44x \end{aligned}$$

c) The volume of this shape = (area of the base)(height).

$$\begin{aligned} Area of the base &= x(x+2) \\ &= x^2+2x \\ \text{Height} &= 2x+1 \\ \text{Volume} &= (x^2+2x)(2x+1) \\ &= 2x^3+x^2+4x^2+2x \\ &= 2x^3+5x^2+2x \end{aligned}$$

d) The volume of this shape = (area of the base)(height).

$$\begin{aligned} Area of the base &= (4a-3)(2a+1) \\ &= 8a^2+4a-6a-3 \\ &= 8a^2-2a-3 \\ \text{Height} &= a+4 \\ \text{Volume} &= (8a^2-2a-3)(a+4) \end{aligned}$$

so the volume is $8a^3+30a^2-11a-12$.

Practice Set

Multiply the following monomials.

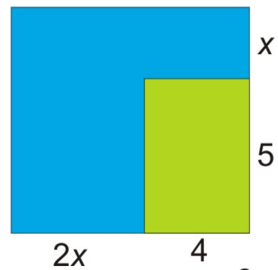
1. $(2x)(-7x)$
2. $(10x)(3xy)$
3. $(4mn)(0.5nm^2)$
4. $(-5a^2b)(-12a^3b^3)$
5. $(3xy^2z^2)(15x^2yz^3)$

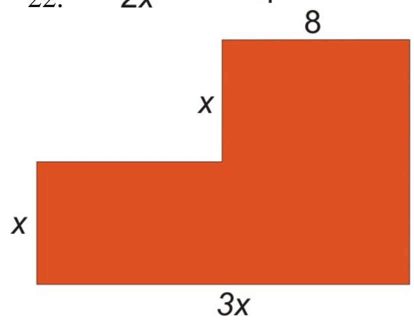
Multiply and simplify.

6. $17(8x-10)$
7. $2x(4x-5)$
8. $9x^3(3x^2-2x+7)$
9. $3x(2y^2+y-5)$
10. $10q(3q^2r+5r)$
11. $-3a^2b(9a^2-4b^2)$
12. $(x-3)(x+2)$
13. $(a+b)(a-5)$
14. $(x+2)(x^2-3)$
15. $(a^2+2)(3a^2-4)$
16. $(7x-2)(9x-5)$
17. $(2x-1)(2x^2-x+3)$
18. $(3x+2)(9x^2-6x+4)$
19. $(a^2+2a-3)(a^2-3a+4)$

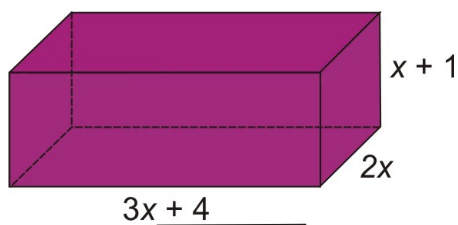
20. $3(x-5)(2x+7)$
21. $5x(x+4)(2x-3)$

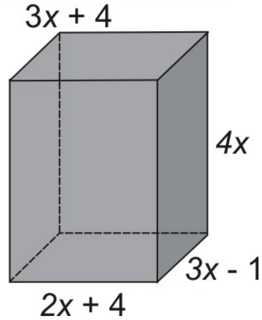
Find the areas of the following figures.

22. 

23. 

Find the volumes of the following figures.

24. 

25. 

3.3 Special Products of Polynomials

We saw that when we multiply two binomials we need to make sure to multiply each term in the first binomial with each term in the second binomial. Let's look at another example.

Multiply two linear binomials (binomials whose degree is 1):

$$(2x+3)(x+4)$$

When we multiply, we obtain a quadratic polynomial (one with degree 2) with four terms:

$$2x^2+8x+3x+12$$

The middle terms are like terms and we can combine them. We simplify and get $2x^2+11x+12$. This is a quadratic, or second-degree, **trinomial** (polynomial with three terms).

You can see that every time we multiply two linear binomials with one variable, we will obtain a quadratic polynomial. In this section we'll talk about some special products of binomials.

Find the Square of a Binomial

One special binomial product is the **square of a binomial**. Consider the product $(x+4)(x+4)$.

Since we are multiplying the same expression by itself, that means we are squaring the expression. $(x+4)(x+4)$ is the same as $(x+4)^2$.

When we multiply it out, we get $x^2+4x+4x+16$, which simplifies to $x^2+8x+16$.

Notice that the two middle terms—the ones we added together to get $8x$—were the same. Is this a coincidence? In order to find that out, let's square a general linear binomial.

$$\begin{aligned}(a+b)^2 = (a+b)(a+b) &= a^2+ab+ab+b^2\\ &= a^2+2ab+b^2\end{aligned}$$

Sure enough, the middle terms are the same. How about if the expression we square is a difference instead of a sum?

$$\begin{aligned}(a-b)^2 = (a-b)(a-b) &= a^2-ab-ab+b^2\\ &= a^2-2ab+b^2\end{aligned}$$

It looks like the middle two terms are the same in general whenever we square a binomial. The general pattern is: to square a binomial, take the square of the first term, add or subtract twice the product of the terms, and add the square of the second term. You should remember these formulas:

$$(a+b)^2 = a^2 + 2ab + b^2$$
and
$$(a-b)^2 = a^2 - 2ab + b^2$$

Remember! Raising a polynomial to a power means that we multiply the polynomial by itself however many times the exponent indicates. For instance, $(a+b)^2 = (a+b)(a+b)$. **Don't make the common mistake of thinking that** $(a+b)^2 = a^2 + b^2$**!** To see why that's not true, try substituting numbers for a and b into the equation (for example, $a = 4$ and $b = 3$), and you will see that it is *not* a true statement. The middle term, $2ab$, is needed to make the equation work.

We can apply the formulas for squaring binomials to any number of problems.

Example 1

Square each binomial and simplify.

a) $(x+10)^2$

b) $(2x-3)^2$

c) $(x^2+4)^2$

d) $(5x-2y)^2$

Solution

Let's use the square of a binomial formula to multiply each expression.

a) $(x+10)^2 =$

$$(x+10)(x+10) = (x)(x) + (x)(10) + (10)(x) + (10)(10) = x^2 + 10x + 10x + 100$$

which simplifies to $x^2 + 20x + 100$.

b) $(2x-3)^2 =$

$$(2x-3)(2x-3) = (2x)(2x) + (2x)(-3) + (-3)(2x) + (-3)(-3) = 4x^2 - 6x - 6x + 9$$

which simplifies to $4x^2 - 12x + 9$.

c) $(x^2+4)^2$

we can multiply this out, or use the formula. If we let $a = x^2$ and $b = 4$, then

$$\begin{aligned}(x^2+4)^2 &= (x^2)^2 + 2(x^2)(4) + (4)^2\\ &= x^4 + 8x^2 + 16\end{aligned}$$

d) $(5x-2y)^2$

If we let $a = 5x$ and $b = 2y$, then

$$\begin{aligned}(5x-2y)^2 &= (5x)^2 - 2(5x)(2y) + (2y)^2\\ &= 25x^2 - 20xy + 4y^2\end{aligned}$$

Find the Product of Binomials Using Sum and Difference Patterns

Another special binomial product is the product of a sum and a difference of terms. For example, let's multiply the following binomials.

$$\begin{aligned}(x+4)(x-4) &= x^2-4x+4x-16\\ &= x^2-16\end{aligned}$$

Notice that the middle terms are opposites of each other, so they *cancel out* when we collect like terms. This is not a coincidence. This always happens when we multiply a sum and difference of the same terms. In general,

$$\begin{aligned}(a+b)(a-b) &= a^2-ab+ab-b^2\\ &= a^2-b^2\end{aligned}$$

When multiplying a sum and difference of the same two terms, the middle terms cancel out. We get the square of the first term minus the square of the second term. You should remember this formula.

Sum and Difference Formula: $(a+b)(a-b) = a^2-b^2$

Let's apply this formula to a few examples.

Example 2

Multiply the following binomials and simplify.

a) $(x+3)(x-3)$

b) $(5x+9)(5x-9)$

c) $(2x^3+7)(2x^3-7)$

d) $(4x+5y)(4x-5y)$

Solution

a) Let $a = x$ and $b = 3$, then:

$$\begin{aligned}(a+b)(a-b) &= a^2-b^2\\ (x+3)(x-3) &= x^2-3^2\\ &= x^2-9\end{aligned}$$

b) Let $a = 5x$ and $b = 9$, then:

$$\begin{aligned}(a+b)(a-b) &= a^2-b^2\\ (5x+9)(5x-9) &= (5x)^2-9^2\\ &= 25x^2-81\end{aligned}$$

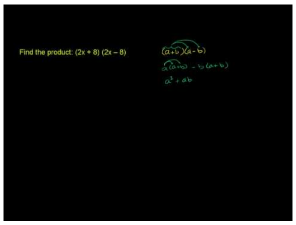
c) Let $a = 2x^3$ and $b = 7$, then:

$$\begin{aligned}(2x^3+7)(2x^3-7) &= (2x^3)^2-(7)^2\\ &= 4x^6-49\end{aligned}$$

d) Let $a = 4x$ and $b = 5y$, then:

$$\begin{aligned}(4x+5y)(4x-5y) &= (4x)^2-(5y)^2\\ &= 16x^2-25y^2\end{aligned}$$

For additional explanation, watch the video on special products of polynomials.



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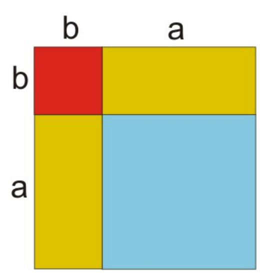
Solve Real-World Problems Using Special Products of Polynomials

Now let's see how special products of polynomials apply to geometry problems and to mental arithmetic.

Example 3

Find the area of the following square:



Solution

The length of each side is $(a+b)$, so the area is $(a+b)(a+b)$.

Notice that this gives a visual explanation of the square of a binomial. The blue square has area a^2, the red square has area b^2, and each rectangle has area ab, so added all together, the area $(a+b)(a+b)$ is equal to $a^2+2ab+b^2$.

The next example shows how you can use the special products to do fast mental calculations.

Example 4

Use the difference of squares and the binomial square formulas to find the products of the following numbers without using a calculator.

a) 43×57

b) 112×88

c) 45^2

d) 481×319

Solution

The key to these mental "tricks" is to rewrite each number as a sum or difference of numbers you know how to square easily.

a) Rewrite 43 as $(50-7)$ and 57 as $(50+7)$.

Then $43 \times 57 = (50-7)(50+7) = (50)^2 - (7)^2 = 2500 - 49 = 2451$

b) Rewrite 112 as $(100+12)$ and 88 as $(100-12)$.

Then $112 \times 88 = (100+12)(100-12) = (100)^2 - (12)^2 = 10000 - 144 = 9856$

c) $45^2 = (40+5)^2 = (40)^2 + 2(40)(5) + (5)^2 = 1600 + 400 + 25 = 2025$

d) Rewrite 481 as $(400+81)$ and 319 as $(400-81)$.

Then $481 \times 319 = (400+81)(400-81) = (400)^2 - (81)^2$

$(400)^2$ is easy - it equals 160000.

$(81)^2$ is not easy to do mentally, so let's rewrite 81 as $80+1$.

$(81)^2 = (80+1)^2 = (80)^2 + 2(80)(1) + (1)^2 = 6400 + 160 + 1 = 6561$

Then $481 \times 319 = (400)^2 - (81)^2 = 160000 - 6561 = 153439$

Practice Set - Using Special Products of Polynomials

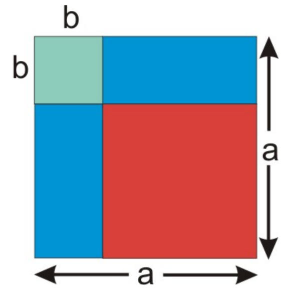
Use the special product rule for squaring binomials to multiply these expressions.

1. $(x+9)^2$
2. $(3x-7)^2$
3. $(5x-y)^2$
4. $(2x^3-3)^2$
5. $(4x^2+y^2)^2$
6. $(8x-3)^2$
7. $(2x+5)(5+2x)$
8. $(xy-y)^2$

Use the special product of a sum and difference to multiply these expressions.

9. $(2x-1)(2x+1)$
10. $(x-12)(x+12)$
11. $(5a-2b)(5a+2b)$
12. $(ab-1)(ab+1)$
13. $(z^2+y)(z^2-y)$
14. $(2q^3+r^2)(2q^3-r^2)$
15. $(7s-t)(t+7s)$
16. $(x^2y+xy^2)(x^2y-xy^2)$

Find the area of the lower right square in the following figure.

17. 

CHAPTER 4 Polynomials and Exponents, Part 2

Chapter Outline

In this chapter, we learn about the rules that apply to dividing terms with exponents, and we learn how to divide simple polynomials.

4.1 Exponential Properties Involving Quotients

In this lesson you will learn how to simplify quotients of numbers and variables.

Quotient of Powers Property: When the exponent in the numerator is larger than the exponent in the denominator, for all real numbers x, $\frac{x^n}{x^m} = x^{n-m}$

When dividing expressions with the same base, keep the base and subtract the exponent in the denominator (bottom) from the exponent in the numerator (top). When we have problems with different bases, we apply the rule separately for each base. To simplify $\frac{x^7}{x^4}$, repeated multiplication can be used.

$$\frac{x^7}{x^4} = \frac{\cancel{x}\cdot\cancel{x}\cdot\cancel{x}\cdot\cancel{x}\cdot x\cdot x\cdot x}{\cancel{x}\cdot\cancel{x}\cdot\cancel{x}\cdot\cancel{x}} = \frac{x\cdot x\cdot x}{1} = x^3$$
$$\frac{x^5y^3}{x^3y^2} = \frac{\cancel{x}\cdot\cancel{x}\cdot\cancel{x}\cdot x\cdot x}{\cancel{x}\cdot\cancel{x}\cdot\cancel{x}}\cdot\frac{\cancel{y}\cdot\cancel{y}\cdot y}{\cancel{y}\cdot\cancel{y}} = \frac{x\cdot x}{1}\cdot\frac{y}{1} = x^2y \text{ OR } \frac{x^5y^3}{x^3y^2} = x^{5-3}\cdot y^{3-2} = x^2y$$

Example 1: *Simplify each of the following expressions using the* ***quotient of powers property.***

(a) $\frac{x^{10}}{x^5}$

(b) $\frac{x^5\gamma^4}{x^3\gamma^2}$

Solution:

(a) $\frac{x^{10}}{x^5} = \chi^{10-5} = \chi^5$

(b) $\frac{x^5y^4}{x^3y^2} = x^{5-3}\cdot y^{4-2} = x^2y^2$

Power of a Quotient Property: $\left(\frac{x^n}{y^m}\right)^p = \frac{x^{n\cdot p}}{y^{m\cdot p}}$

The power inside the parenthesis for the numerator and the denominator multiplies with the power outside the parenthesis. The situation below shows why this property is true.

$$\left(\frac{x^3}{y^2}\right)^4 = \left(\frac{x^3}{y^2}\right)\cdot\left(\frac{x^3}{y^2}\right)\cdot\left(\frac{x^3}{y^2}\right)\cdot\left(\frac{x^3}{y^2}\right) = \frac{(x\cdot x\cdot x)\cdot(x\cdot x\cdot x)\cdot(x\cdot x\cdot x)\cdot(x\cdot x\cdot x)}{(y\cdot y)\cdot(y\cdot y)\cdot(y\cdot y)\cdot(y\cdot y)} = \frac{x^{12}}{y^8}$$

Example 2: *Simplify the following expression.*

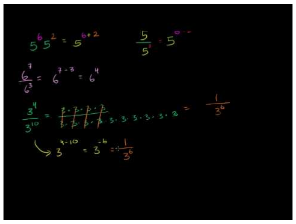
$$\left(\frac{x^{10}}{y^5}\right)^3$$

Solution: $\left(\frac{x^{10}}{y^5}\right)^3 = \frac{x^{10\cdot 3}}{y^{5\cdot 3}} = \frac{x^{30}}{y^{15}}$

Practice Set: Exponents Involving Quotients

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the

practice exercise listed in the following exercise set. However, the practice exercise is the same in both. Exponent Properties Involving Quotients (9:22)



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Evaluate the following expressions.

1. $\frac{5^6}{5^2}$
2. $\frac{6^7}{6^3}$
3. $\frac{3^{10}}{3^4}$
4. $\left(\frac{2^2}{3^3}\right)^3$

Simplify the following expressions.

1. $\frac{a^3}{a^2}$
2. $\frac{x^9}{x^5}$
3. $\frac{x^{10}}{x^5}$
4. $\frac{a^6}{a}$
5. $\frac{a^5b^4}{a^3b^2}$
6. $\frac{4^5}{4^2}$
7. $\left(\frac{3^4}{5^2}\right)^2$
8. $\left(\frac{a^3b^4}{a^2b}\right)^3$
9. $\frac{x^6y^5}{x^2y^3}$
10. $\frac{6x^2y^3}{2xy^2}$
11. $\left(\frac{2a^3b^3}{8a^7b}\right)^2$
12. $(x^2)^2 \cdot \frac{x^6}{x^4}$
13. $\left(\frac{16a^2}{4b^5}\right)^3 \cdot \frac{b^2}{a^{16}}$
14. $\frac{6a^3}{2a^2}$
15. $\frac{15x^5}{5x}$
16. $\left(\frac{18a^{10}}{15a^4}\right)^4$
17. $\frac{25yx^6}{20y^5x^2}$

18. $\left(\frac{x^6y^2}{x^4y^4}\right)^3$
19. $\left(\frac{6a^2}{4b^4}\right)^2 \cdot \frac{5b}{3a}$
20. $\frac{(3ab)^2(4a^3b^4)^3}{(6a^2b)^4}$
21. $\frac{(2a^2bc^2)(6abc^3)}{4ab^2c}$

4.2 Exponential Properties Involving Zero and Negative Exponents

In the previous lessons we have dealt with powers that are positive whole numbers. In this lesson, you will learn how to simplify expressions when the exponent is zero, or negative.

Exponents of Zero: for all real numbers of $x, x \neq 0, x^0 = 1$

Example: $1 = \frac{x^4}{x^4} = x^{4-4} = x^0$. This example is simplified using the Quotient of Powers Property. We can also see why we make the restriction that $x \neq 0$, since $\frac{0^4}{0^4}$ would require us to divide by zero which is never allowed. Thus, 0^0 is not defined.

Simplifying Expressions with Negative Exponents

The next objective is **negative exponents.** When we use the quotient rule and we subtract a greater number from a smaller number, the answer will become negative. The variable and the power will be moved to the denominator of a fraction. You will learn how to write this in an expression.

Example: $\frac{x^4}{x^6} = x^{4-6} = x^{-2}$. If we write the expression in expanded form, we have $\frac{\chi \cdot \chi \cdot \chi \cdot \chi}{\chi \cdot \chi \cdot \chi \cdot \chi \cdot \chi \cdot \chi}$. The four x's on top will cancel out with four x's on bottom. This will leave 2 x's remaining on the bottom which makes your answer look like $\frac{1}{x^2}$. So $x^{-2} = \frac{1}{x^2}$. In general:

Negative Power rule for Exponents: $\frac{1}{x^n} = x^{-n}$ where $\chi \neq 0$

Example: $x^{-6}y^{-2} = \frac{1}{x^6} \cdot \frac{1}{y^2} = \frac{1}{x^6y^2}$. The negative power rule for exponents is applied to both variables separately in this example.

Multimedia Link: For more help with these types of exponents, watch this **http://tinyurl.com/7n3ae2l**.

Example 1: *Write the following expressions without fractions.*

(a) $\frac{2}{x^2}$

(b) $\frac{x^2}{y^3}$

Solution:

(a) $\frac{2}{x^2} = 2x^{-2}$

(b) $\frac{x^2}{y^3} = x^2y^{-3}$

Notice example 1a, the number 2 is in the numerator. This number is multiplied to x^{-2}. It could also look like this, $2 \cdot \frac{1}{x^2}$ to be better understood.

It is important when evaluating expressions you remember the order of operations. Evaluate what is inside the parentheses or other grouping symbols, then evaluate the exponents, then perform multiplication/division from left to right, then perform addition/subtraction from left to right.

Example 2: *Evaluate the following expression*

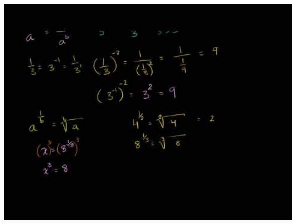
(a) $3 \cdot 5^2 - 10 \cdot 5 + 1$

Solution: $3 \cdot 5^2 - 10 \cdot 5 + 1 = 3 \cdot 25 - 10 \cdot 5 + 1 = 75 - 50 + 1 = 26$

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the

practice exercise listed in the following exercise set. However, the practice exercise is the same in both. Zeroand Negative Exponents (14:04)



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Simplify the following expressions. Be sure the final answer includes only positive exponents.

1. $x^{-1} \cdot y^2$
2. x^{-4}
3. $\frac{x^{-3}}{x^{-7}}$
4. $\frac{1}{x}$
5. $\frac{2}{x^2}$
6. $\frac{x^2}{y^3}$
7. $\frac{3}{xy}$
8. $3x^{-3}$
9. $a^2b^{-3}c^{-1}$
10. $4x^{-1}y^3$
11. $\frac{2x^{-2}}{y^{-3}}$
12. $\frac{x^{-3}y^{-5}}{z^{-7}}$
13. $\left(\frac{a}{b}\right)^{-2}$
14. $(3a^{-2}b^2c^3)^3$
15. $x^{-3} \cdot x^3$

Simplify the following expressions without any fractions in the answer.

16. $\frac{a^{-3}(a^5)}{a^{-6}}$

17. $\frac{5x^6y^2}{x^8y}$

18. $\frac{(4ab^6)^3}{(ab)^5}$

19. $\frac{4a^2b^3}{2a^5b}$

20. $\left(\frac{x}{3y^2}\right)^3 \cdot \frac{x^2y}{4}$

21. $\left(\frac{ab^{-2}}{b^3}\right)^2$

22. $\frac{x^{-3}y^2}{x^2y^{-2}}$

23. $\frac{(3x^3)(4x^4)}{(2y)^2}$

24. $\frac{a^{-2}b^{-3}}{c^{-1}}$

Evaluate the following expressions to a single number.

25. 3^{-2}

26. $(6.2)^0$

27. $8^{-4} \cdot 8^6$

28. 5^0

29. 7^2

30. $\left(\frac{2}{3}\right)^3$

31. 3^{-3}

Evaluate the following expressions for $x = 2, y = -1, z = 3$.

32. $2x^2 - 3y^3 + 4z$

33. $(x^2 - y^2)^2$

34. $\left(\frac{3x^2y^5}{4z}\right)^{-2}$

35. $x^2 4x^3 y^4 4y^2$ if $x = 2$ and $y = -1$

36. $a^4(b^2)^3 + 2ab$ if $a = -2$ and $b = 1$

37. $5x^2 - 2y^3 + 3z$ if $x = 3,\ y = 2$, and $z = 4$

38. $\left(\frac{a^2}{b^3}\right)^{-2}$ if $a = 5$ and $b = 3$

39. $3 \cdot 5^5 - 10 \cdot 5 + 1$

40. $\frac{2 \cdot 4^2 - 3 \cdot 5^2}{3^2}$

4.3 Division of Polynomials

When we take the ratio of two polynomials, we call the result a **rational expression.** We can also think of this as division of polynomials.

Some examples of rational expressions are

$$\frac{x^2-x}{x} \qquad \frac{4x^2-3x+4}{2x}$$

Just as with rational numbers, the expression on the top is called the **numerator** and the expression on the bottom is called the **denominator**. In special cases we can simplify a rational expression by dividing the numerator by the denominator.

Divide a Polynomial by a Monomial

We'll start by dividing a polynomial by a monomial. To do this, we divide each term of the polynomial by the monomial. When the numerator has more than one term, the monomial on the bottom of the fraction serves as the **common** denominator to all the terms in the numerator.

Example 1

Divide.

a) $\frac{8x^2-4x+16}{2}$

b) $\frac{3x^2+6x-1}{x}$

c) $\frac{-3x^2-18x+6}{9x}$

Solution

a) $\frac{8x^2-4x+16}{2} = \frac{8x^2}{2} - \frac{4x}{2} + \frac{16}{2} = 4x^2 - 2x + 8$

b) $\frac{3x^3+6x-1}{x} = \frac{3x^3}{x} + \frac{6x}{x} - \frac{1}{x} = 3x^2 + 6 - \frac{1}{x}$

c) $\frac{-3x^2-18x+6}{9x} = -\frac{3x^2}{9x} - \frac{18x}{9x} + \frac{6}{9x} = -\frac{x}{3} - 2 + \frac{2}{3x}$

A common error is to cancel the denominator with just one term in the numerator.

Consider the quotient $\frac{3x+4}{4}$.

Remember that the denominator of 4 is common to both the terms in the numerator. In other words we are dividing both of the terms in the numerator by the number 4.

The correct way to simplify is:

$$\frac{3x+4}{4} = \frac{3x}{4} + \frac{4}{4} = \frac{3x}{4} + 1$$

A common mistake is to cross out the number 4 from the numerator and the denominator, leaving just $3x$. This is incorrect, because the entire numerator needs to be divided by 4.

Example 2

Divide $\frac{5x^3-10x^2+x-25}{-5x^2}$.

Solution

$$\frac{5x^3-10x^2+x-25}{-5x^2}=\frac{5x^3}{-5x^2}-\frac{10x^2}{-5x^2}+\frac{x}{-5x^2}-\frac{25}{-5x^2}$$

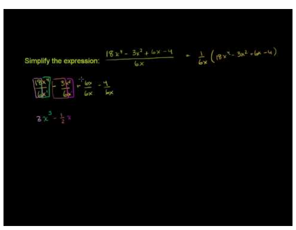
The negative sign in the denominator changes all the signs of the fractions:

$$-\frac{5x^3}{5x^2}+\frac{10x^2}{5x^2}-\frac{x}{5x^2}+\frac{25}{5x^2}=-x+2-\frac{1}{5x}+\frac{5}{x^2}$$

Simplify the expression:

$$\frac{18x^3-3x^2+6x-4}{6x}$$

Watch the video explanation toseethis example workedout.



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Practice Set - Division of Polynomials

Divide the following polynomials:

1. $\frac{2x+4}{2}$
2. $\frac{x-4}{x}$
3. $\frac{5x-35}{5x}$
4. $\frac{x^2+2x-5}{x}$
5. $\frac{4x^2+12x-36}{-4x}$
6. $\frac{2x^2+10x+7}{2x^2}$
7. $\frac{x^3-x}{-2x^2}$
8. $\frac{5x^4-9}{3x}$
9. $\frac{x^3-12x^2+3x-4}{12x^2}$
10. $\frac{3-6x+x^3}{-9x^3}$
11. $\frac{x^2-6x-12}{5x^4}$

CHAPTER 5 Solving Equations

Chapter Outline

Mathematical equations are used in many different career fields. Medical researchers use equations to determine the length of time it takes for a drug to circulate throughout the body, botanists use equations to determine the amount of time it takes a Sequoia tree to reach a particular height, and environmental scientists can use equations to approximate the number of years it will take to repopulate the bison species.

In this chapter, you will learn how to manipulate linear equations to solve for the unknown quantity represented by the variable. You already have some experience solving equations. This chapter is designed to help formalize the mental math you use to answer questions in daily life.

5.1 The Solution of an Equation

When an algebraic expression is set equal to another value, variable, or expression, a new mathematical sentence is created. This sentence is called an **equation.**

Definition: An **algebraic equation** is a mathematical sentence connecting an expression to a value, variable, or another expression with an equal sign (=).

Definition: The **solution** to an equation is the value (or multiple values) that make the equation true..

What is the value of m in the following equation?

$$\frac{1}{4}m = 20.00$$

Think: One-quarter can also be thought of as *divide by four.* What divided by 4 equals 20.00?

The solution is 80. So, the money was $80.00.

Checking an answer to an equation is almost as important as the equation itself. By substituting the value for the variable, you are making sure both sides of the equation balance.

Example 1: Check that $x = 5$ is the solution to the equation $3x+2 = -2x+27$.

Solution: To check that $x = 5$ is the solution to the equation, substitute the value of 5 for the variable, x:

$$\begin{aligned}
3x+2 &= -2x+27\\
3\cdot x+2 &= -2\cdot x+27\\
3\cdot 5+2 &= -2\cdot 5+27\\
15+2 &= -10+27\\
17 &= 17
\end{aligned}$$

Because $17 = 17$ is a true statement, we can conclude that $x = 5$ is a solution to $3x+2 = -2x+27$.

Example 2: Is $z = 3$ a solution to $z^2+2z = 8$?

Solution: Begin by substituting the value of 3 for z.

$$\begin{aligned}
3^2+2(3) &= 8\\
9+6 &= 8\\
15 &= 8
\end{aligned}$$

Because $15 = 8$ is NOT a true statement, we can conclude that $z = 3$ is not a solution to $z^2+2z = 8$.

Practice Problems

Check that the given number is a solution to the corresponding equation.

1. $a = -3$; $4a+3 = -9$

2. $x = \frac{4}{3};\ \frac{3}{4}x + \frac{1}{2} = \frac{3}{2}$
3. $y = 2;\ 2.5y - 10.0 = -5.0$
4. $z = -5;\ 2(5 - 2z) = 20 - 2(z - 1)$

5.2 One-Step Equations

You have been solving equations since the beginning of this textbook although you may not have recognized it. For example, in a previous lesson, you determined the answer to the pizza problem below.

$20.00 *was one-quarter of the money spent on pizza.*

$\frac{1}{4}m = 20.00$ What divided by 4 equals 20.00?

The solution is 80. So, the amount of money before buying the pizza was $80.00.

By working through this question mentally, you were applying mathematical rules and solving for the variable m.

Definition: To **solve** an equation means to write an equivalent equation that has the variable by itself on one side. This is also known as **isolating the variable.**

In order to begin solving equations, you must understand three basic concepts of algebra: inverse operations, equivalent equations, and the Addition Property of Equality.

Inverse Operations and Equivalent Equations

In lesson 1, you learned how to simplify an expression using the order of operations: **P**arentheses, **E**xponents, **M**ultiplication and **D**ivision completed in order from left to right, and **A**ddition and **S**ubtraction (also completed from left to right). Each of these operations has an **inverse.** Inverse operations "undo" each other when combined.

For example, the inverse of addition is subtraction. The inverse of an exponent is a root.

Example 1: *Determine the inverse of division.*

Solution: To undo dividing something, you would multiply.

By applying the same inverse operations to each side of an equation, you create an **equivalent equation**.

Definition: Equivalent equations are two or more equations having the same solution.

The Addition Property of Equality

Just like Spanish, Chemistry, or even music, mathematics has a set of rules you must follow in order to be successful. These are called properties, theorems, or axioms. These have been proven or agreed upon years ago so you can apply them to many different situations.

For example, **The Addition Property of Equality** allows you to apply the same operation to each side of the equation, or what you do to one side of an equation you can do to the other.

The Addition Property of Equality

For all real numbers a, b, and c:

If $a = b$, then $a + c = b + c$.

Solving One-Step Equations Using Addition or Subtraction

Because subtraction can be considered adding a negative, the Addition Property of Equality also works if you need to subtract the same value from each side of an equation.

Example 2

Solve for y : $16 = y - 11$.

Solution: When asked to solve for y, your goal is to write an equivalent equation with the variable y isolated on one side.

Write the original equation $16 = y - 11$.

Apply the Addition Property of Equality $16 + 11 = y - 11 + 11$

Simplify by adding like terms $27 = y$.

The solution is $y = 27$.

Example 3: *One method to weigh a horse is to load it into an empty trailer with a known weight and reweigh the trailer. A Shetland pony is loaded onto a trailer that weighs 2,200 pounds empty and re-weighed. The new weight is 2,550 pounds. How much does the pony weigh?*

Solution: Choose a variable to represent the weight of the pony, say p.

Write an equation $2550 = 2200 + p$.

Apply the Addition Property of Equality $2550 - 2200 = 2200 + p - 2200$

Simplify $350 = p$.

The Shetland pony weighs 350 pounds.

Equations that take one step to isolate the variable are called **one-step equations**. Such equations can also involve multiplication or division.

The Multiplication Property of Equality

The Multiplication Property of Equality says that if you multiply one side of an equation by a number, you will maintain the equality if you also multiply the other side of that equation by the number.

For all real numbers a, b, and c:

If $a = b$, *then* $a(c) = b(c)$

Since division is like multiplying by the reciprocal, the Multiplication Property of Equality also works if you divide each side of the equation by the same number.

Solving One-Step Equations Using Multiplication or Division

Example 4

Solve for k : $-8k = -96$

Solution: Because $-8k = -8 \times k$, the inverse operation of multiplication is division. Therefore, we must cancel multiplication by applying the Multiplication Property of Equality.

Write the original equation $-8k = -96$.

Apply the Multiplication Property of Equality $-8k \div -8 = -96 \div -8$

The solution is $k = 12$.

When working with fractions, you must remember: $\frac{a}{b} \times \frac{b}{a} = 1$. In other words, in order to cancel a fraction using division, you really must multiply by its reciprocal.

Example 5: *Solve* $\frac{1}{8} \cdot x = 1.5$.

The variable x is being multiplied by one-eighth. Instead of dividing two fractions, we multiply by the reciprocal of $\frac{1}{8}$, which is 8.

$$\cancel{8}\left(\frac{1}{\cancel{8}} \cdot x\right) = 8(1.5)$$
$$x = 12$$

Solving Real World Problems Using Equations

As was mentioned in the chapter opener, many careers base their work on manipulating linear equations. Consider the botanist studying bamboo as a renewable resource. She knows bamboo can grow up to 60 centimeters per day. If the specimen she measured was 1 meter tall, how long would it take to reach 5 meters in height? By writing and solving this equation, she will know exactly how long it should take for the bamboo to reach the desired height.

Example 6: In good weather, tomato seeds can grow into plants and bear ripe fruit in as little as 19 weeks. Lorna planted her seeds 11 weeks ago. How long must she wait before her tomatoes are ready to be picked?

Solution: The variable in question is the number of weeks until the tomatoes are ready. Call this variable w.

Write an equation $w + 11 = 19$

Solve for w by using the Addition Property of Equality.

$$w + 11 - 11 = 19 - 11$$
$$w = 8$$

It will take as little as 8 weeks for the plant to bear ripe fruit.

Example 7: *In 2004, Takeru Kobayashi, of Nagano, Japan, ate* $53\frac{1}{2}$ *hot dogs in 12 minutes. He broke his previous world record, set in 2002, by three more hot dogs. Calculate how many minutes it took him to eat one hot dog.*

Solution:

Letting m represent the number of minutes to eat one hot dog. Then, $53.5m = 12$

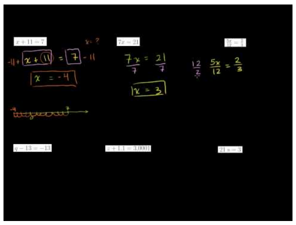
Applying the Multiplication Property of Equality,

$$\frac{53.5m}{53.5} = \frac{12}{53.5}$$
$$m = 0.224 \ minutes$$

It took approximately 0.224 minutes or 13.5 seconds to eat one hot dog.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. CK-12 Basic Algebra:One-Step Equations (12:30)



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Solve for the given variable.

1. $x+11=7$
2. $x-1.1=3.2$
3. $7x=21$
4. $4x=1$
5. $\frac{5x}{12}=\frac{2}{3}$
6. $x+\frac{5}{2}=\frac{2}{3}$
7. $x-\frac{5}{6}=\frac{3}{8}$
8. $0.01x=11$
9. $q-13=-13$
10. $z+1.1=3.0001$
11. $21\,s=3$
12. $t+\frac{1}{2}=\frac{1}{3}$
13. $\frac{7f}{11}=\frac{7}{11}$
14. $\frac{3}{4}=-\frac{1}{2}\cdot y$
15. $6r=\frac{3}{8}$
16. $\frac{9b}{16}=\frac{3}{8}$
17. Peter is collecting tokens on breakfast cereal packets in order to get a model boat. In eight weeks he has collected 10 tokens. He needs 25 tokens for the boat. Write an equation and determine the following information.
    (a) How many more tokens he needs to collect, n.
    (b) How many tokens he collects per week, w.
    (c) How many more weeks remain until he can send off for his boat, r.
18. Juan has baked a cake and wants to sell it in his bakery. He is going to cut it into 12 slices and sell them individually. He wants to sell it for three times the cost of making it. The ingredients cost him \$8.50, and he allowed \$1.25 to cover the cost of electricity to bake it. Write equations that describe the following statements.
    (a) The amount of money that he sells the cake for (u).
    (b) The amount of money he charges for each slice (c).

(c) The total profit he makes on the cake (w).

19. Solve the remaining two questions regarding Takeru Kobayashi.

5.3 Two-Step Equations

Suppose Shaun weighs 146 pounds and wants to lose enough weight to wrestle in the 130-pound class. His nutritionist designed a diet for Shaun so he will lose about 2 pounds per week. How many weeks will it take Shaun to weigh enough to wrestle in his class?

This is an example which can be solved by working backward. In fact, you may have already found the answer by using this method. *The solution is 8 weeks*.

By translating this situation into an algebraic sentence, we can begin the process of **solving equations**. To solve an equation means to undo all the operations of the sentence, leaving a value for the variable.

Translate Shauns situation into an equation:

$$-2w+146=130$$

This sentence has two operations: addition and multiplication. To find the value of the variable, we must use both properties of Equality: The Addition Property of Equality and the Multiplication Property of Equality.

Procedure to Solve Equations of the Form $ax+b=$***some number:***

1. Use the Addition Property of Equality to get the variable term ax alone on one side of the equation:

$$ax = \textit{some number}$$

2. Use the Multiplication Property of Equality to get the variable x alone on one side of the equation:

$$x = \textit{some number}$$

Example 1: *Solve Shauns problem.*

Solution: $-2w+146=130$

Apply the Addition Property of Equality: $-2w+146-146=130-146$

Simplify: $-2w=-16$

Apply the Multiplication Property of Equality: $-2w \div -2 = -16 \div -2$

The solution is $w=8$.

It will take 8 weeks for Shaun to weigh 130 pounds.

Solving Equations by Combining Like Terms

Michigan has a 6% sales tax. Suppose you made a purchase and paid $95.12, including tax. How much was the purchase before tax?

Begin by determining the noun that is unknown and choose a letter as its representation.

The purchase price is unknown so this is our variable. Call it p. Now translate the sentence into an algebraic equation.

$$\begin{aligned} price + (0.06)price &= total\ amount \\ p + 0.06p &= 95.12 \end{aligned}$$

To solve this equation, you must know how to **combine like terms.**

Like terms are expressions that have **identical** variable parts.

According to this definition, you can only combine like terms if they are identical. **Combining like terms only applies to addition and subtraction!** This is not a true statement when referring to multiplication and division.

The numerical part of an algebraic term is called the **coefficient**. To combine like terms, you add (or subtract) the coefficients of the identical variable parts.

Combine the like terms: $p + 0.06p = 1.06p$, since $p = 1p$

Simplify: $1.06p = 95.12$

Apply the Multiplication Property of Equality: $1.06p \div 1.06 = 95.12 \div 1.06$

Simplify: $p = 89.74$

The price before tax was \$89.74.

The next several examples show how algebraic equations can be created to solve real world situations.

Example 1: *An emergency plumber charges* \$65 *as a call-out fee plus an additional* \$75 per hour. He arrives at a house at 9:30 and works to repair a water tank. If the total repair bill is \$196.25, at what time was the repair completed?

Solution: Translate the sentence into an equation. The number of hours it took to complete the job is unknown, so call it h.

Write the equation: $65 + 75(h) = 196.25$

Apply the Addition Property and simplify:

$$\begin{aligned} 65 + 75(h) - 65 &= 196.25 - 65 \\ 75(h) &= 131.25 \end{aligned}$$

Simplify: $h = 1.75$

The plumber worked for 1.75 hours, or 1 hour, 45 minutes. Since he started at 9:30, the repair was completed at 11:15.

Example 2: *To determine the temperature in Fahrenheit, multiply the Celsius temperature by 1.8 then add 32. Determine the Celsius temperature if it is* $89^\circ F$.

Solution: Translate the sentence into an equation. The temperature in Celsius is unknown; call it C.

Write the equation: $1.8C+32=89$

Apply the Addition Property and simplify:

$$\begin{aligned} 1.8C+32-32 &= 89-32 \\ 1.8C &= 57 \end{aligned}$$

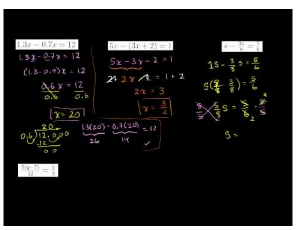
Apply Multiplication Property of Equality: $1.8C \div 1.8 = 57 \div 1.8$

Simplify: $C = 31.67$

If the temperature is $89^\circ F$, then it is $31.67^\circ C$.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. CK-12 Basic Algebra:Two-Step Equations (13:50)



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Solve and check your solution.

1. $1.3x-0.7x=12$
2. $6x-1.3=3.2$
3. $5x-(3x+2)=1$
4. $4(x+3)=1$
5. $5q-7=\frac{2}{3}$
6. $\frac{3}{5}x+\frac{5}{2}=\frac{2}{3}$
7. $s-\frac{3s}{8}=\frac{5}{6}$
8. $0.1y+11=0$
9. $\frac{5q-7}{12}=\frac{2}{3}$
10. $\frac{5(q-7)}{12}=\frac{2}{3}$
11. $33t-99=0$
12. $5p-2=32$
13. $14x+9x=161$

14. $3m-1+4m=5$
15. $8x+3=11$
16. $24=2x+6$
17. $66=\frac{2}{3}k$
18. $\frac{5}{8}=\frac{1}{2}(a+2)$
19. $16=-3d-5$
20. Jayden purchased a new pair of shoes. Including a 7% sales tax, he paid $84.68. How much did his shoes cost before sales tax?
21. A mechanic charges $98 for parts and $60 per hour for labor. Your bill totals $498.00, including parts and labor. How many hours did the mechanic work?
22. An electric guitar and amp set costs $1195.00. You are going to pay $250 as a down payment and pay the rest in 5 equal installments. How much should you pay each month?
23. Jade is stranded downtown with only $10 to get home. Taxis cost $0.75 per mile, but there is an additional $2.35 hire charge. Write a formula and use it to calculate how many miles she can travel with her money. Determine how many miles she can ride.
24. Jasmins Dad is planning a surprise birthday party for her. He will hire a bouncy castle, and will provide party food for all the guests. The bouncy castle costs $150 dollars for the afternoon, and the food will cost $3.00 per person. Andrew, Jasmins Dad, has a budget of $300. Write an equation to help him determine the maximum number of guests he can invite.

5.4 Multi-Step Equations

So far in this chapter you have learned how to solve one-step equations of the form $y = ax$ and two-step equations of the form $y = ax + b$. This lesson will expand upon solving equations to include solving multi-step equations and equations involving the Distributive Property.

Solving Multi-Step Equations by Combining Like Terms

In the last lesson, you learned the definition of like terms and how to combine such terms. We will use the following situation to further demonstrate solving equations involving like terms.

You are hosting a Halloween party. You will need to provide 3 cans of soda per person, 4 slices of pizza per person, and 37 party favors. You have a total of 79 items. How many people are coming to your party?

This situation has several pieces of information: soda cans, slices of pizza, and party favors. Translate this into an algebraic equation.

$$3p + 4p + 37 = 79$$

This equation requires three steps to solve. In general, to solve any equation you should follow this procedure.

Procedure to Solve Equations:

1. Remove any parentheses by using the Distributive Property or the Multiplication Property of Equality:

2. Simplify each side of the equation by combining like terms.

3. Isolate the ax term. Use the Addition Property of Equality to get the variable on one side of the equal sign and the numerical values on the other.

4. Isolate the variable. Use the Multiplication Property of Equality to get the variable alone on one side of the equation.

5. Check your solution.

Example 1: *Determine the number of party-goers in the opening example.*

Solution: $3p + 4p + 37 = 79$

Combine like terms: $7p + 37 = 79$

Apply the Addition Property of Equality: $7p + 37 - 37 = 79 - 37$

Simplify: $7p = 42$

Multiplication Property of Equality: $7p \div 7 = 42 \div 7$

The solution is $p = 6$.

There are six people coming to the party.

Solving Multi-Step Equations by Using the Distributive Property

When faced with an equation such as $2(5x + 9) = 78$, the first step is to remove the parentheses. There are two options to remove the parentheses. You can apply the Distributive Property or you can apply the Multiplication Property of Equality. This lesson will show you how to use the Distributive Property to solve multi-step equations.

Example 2: *Solve for* $x: 2(5x+9) = 78$

Solution: Apply the Distributive Property: $10x+18 = 78$

Apply the Addition Property of Equality: $10x+18-18 = 78-18$

Simplify: $10x = 60$

Multiplication Property of Equality: $10x \div 10 = 60 \div 10$

The solution is $x = 6$.

Check: Does $10(6)+18 = 78$? Yes, so the answer is correct.

Example 3: *Kashmir needs to fence in his puppy. He will fence in three sides, connecting it to his back porch. He wants the run to be 12 feet long and he has 40 feet of fencing. How wide can Kashmir make his puppy enclosure?*

Solution: Translate the sentence into an algebraic equation. Let w represent the width of the enclosure.

$$w+w+12 = 40$$

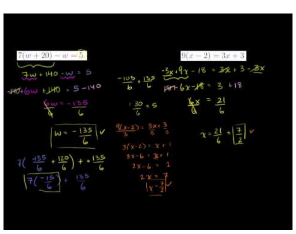
Solve for w:

$$\begin{aligned}2w+12 &= 40\\ 2w+12-12 &= 40-12\\ 2w &= 28\\ 2w \div 2 &= 28 \div 2\\ w &= 14\end{aligned}$$

The dimensions of the enclosure are 14 feet wide by 12 feet long.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. CK-12 Basic Algebra:Multi-Step Equations (15:01)



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Solve for the given variable.

1. $3(x-1)-2(x+3)=0$
2. $7(w+20)-w=5$
3. $9(x-2)=3x+3$
4. $2\left(5a-\frac{1}{3}\right)=\frac{2}{7}$
5. $\frac{2}{9}\left(i+\frac{2}{3}\right)=\frac{2}{5}$
6. $4\left(v+\frac{1}{4}\right)=\frac{35}{2}$
7. $22=2(p+2)$
8. $-(m+4)=-5$
9. $48=4(n+4)$
10. $\frac{6}{5}\left(v-\frac{3}{5}\right)=\frac{6}{25}$
11. $-10(b-3)=-100$
12. $6v+6(4v+1)=-6$
13. $-46=-4(3s+4)-6$
14. $8(1+7m)+6=14$
15. $0=-7(6+3k)$
16. $35=-7(2-x)$
17. $-3(3a+1)-7a=-35$
18. $-2\left(n+\frac{7}{3}\right)=-\frac{14}{3}$
19. $-\frac{59}{60}=\frac{1}{6}\left(-\frac{4}{3}r-5\right)$
20. $\frac{4y+3}{7}=9$
21. $(c+3)-2c-(1-3c)=2$
22. $5m-3[7-(1-2m)]=0$
23. $f-1+2f+f-3=-4$
24. Find four consecutive even integers whose sum is 244.
25. Four more than two-thirds of a number is 22. What is the number?
26. The total cost of lunch is \$3.50, consisting of a juice, a sandwich, and a pear. The juice cost 1.5 times as much as the pear. The sandwich costs \$1.40 more than the pear. What is the price of the pear?
27. Camden High has five times as many desktop computers as laptops. The school has 65 desktop computers. How many laptops does it have?
28. A realtor receives a commission of \$7.00 for every \$100 of a homes selling price. How much was the selling price of a home if the realtor earned \$5,389.12 in commission?

5.5 Equations with Variables on Both Sides

As you may now notice, equations come in all sizes and styles. There are single-step, double-step, and multi-step equations. In this lesson, you will learn how to solve equations with a variable appearing on each side of the equation. The process you need to solve this type of equation is similar to solving a multi-step equation. The procedure is repeated here.

Procedure to Solve Equations:

1. Remove any parentheses by using the Distributive Property or the Multiplication Property of Equality:

2. Simplify each side of the equation by combining like terms.

3. Isolate the *ax* term. Use the Addition Property of Equality to get the variable on one side of the equal sign and the numerical values on the other.

4. Isolate the variable. Use the Multiplication Property of Equality to get the variable alone on one side of the equation.

5. Check your solution.

Karen and Sarah have bank accounts. Karen has a starting balance of $125.00 and is depositing $20 each week. Sarah has a starting balance of $43 and is depositing $37 each week. When will the girls have the same amount of money?

To solve this problem, you could use the guess and check method. You are looking for a particular week in which the bank accounts are equal. This could take a long time! You could also translate the sentence into an equation. The number of weeks is unknown so this is our variable, call it w. Now translate this situation into an algebraic equation:

$$125+20w=43+37w$$

This is a situation in which the variable w appears on both sides of the equation. To begin to solve for the unknown, we must use the Addition Property of Equality to gather the variables on one side of the equation.

Example 1: *Determine when Sarah and Karen will have the same amount of money.*

Solution: Using the Addition Property of Equality, move the variables to one side of the equation:

$$125+20w-20w=43+37w-20w$$

Simplify: $125 = 43 + 17w$

Solve:

$$\begin{aligned} 125 - 43 &= 43 - 43 + 17w \\ 82 &= 17w \\ 82 \div 17 &= 17w \div 17 \\ w &\approx 4.82 \end{aligned}$$

It will take about 4.8 weeks for Sarah and Karen to have equal amounts of money.

Example 2: *Solve for* $h : 3(h+1) = 11h - 23$

Solution: First you must remove the parentheses by using the Distributive Property:

$$3h + 3 = 11h - 23$$

Gather the variables on one side:

$$3h - 3h + 3 = 11h - 3h - 23$$

Simplify:

$$3 = 8h - 23$$

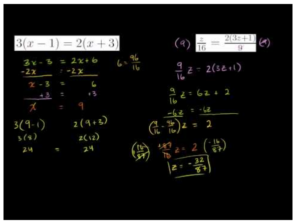
Solve using the steps from lesson 3.3:

$$\begin{aligned} 3 + 23 &= 8h - 23 + 23 \\ 26 &= 8h \\ 26 \div 8 &= 8h \div 8 \\ h &= \frac{13}{4} = 3.25 \end{aligned}$$

Multimedia Link: Watch this video - http://www.teachertube.com/viewVideo.php?video_id=55491&title=Solving_equations_with_variables_on_both_sides for further information on how to solve an equation with a variable on each side of the equation.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. Equations with Variableson BothSides (9:28)



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Solve for the given variable.

1. $3(x-1) = 2(x+3)$
2. $7(x+20) = x+5$
3. $9(x-2) = 3x+3$
4. $2\left(a-\frac{1}{3}\right) = \frac{2}{5}\left(a+\frac{2}{3}\right)$
5. $\frac{2}{7}\left(t+\frac{2}{3}\right) = \frac{1}{5}\left(t-\frac{2}{3}\right)$
6. $\frac{1}{7}\left(v+\frac{1}{4}\right) = 2\left(\frac{3v}{2}-\frac{5}{2}\right)$
7. $\frac{y-4}{11} = \frac{2}{5} \cdot \frac{2y+1}{3}$
8. $\frac{z}{16} = \frac{2(3z+1)}{9}$
9. $\frac{q}{16} + \frac{q}{6} = \frac{(3q+1)}{9} + \frac{3}{2}$
10. $21+3b = 6-6(1-4b)$
11. $-2x+8 = 8(1-4x)$
12. $3(-5v-4) = -6v-39$
13. $-5(5k+7) = 25+5k$
14. Manoj and Tamar are arguing about how a number trick they heard goes. Tamar tells Andrew to think of a number, multiply it by five and subtract three from the result. Then Manoj tells Andrew to think of a number add five and multiply the result by three. Andrew says that whichever way he does the trick he gets the same answer. What was Andrew's number?
15. I have enough money to buy five regular priced CDs and have \$6 left over. However all CDs are on sale today, for \$4 less than usual. If I borrow \$2, I can afford nine of them. How much are CDs on sale for today?
16. Jaime has a bank account with a balance of \$412 and is saving \$18 each week. George has a bank account with a balance of \$874 and is spending \$44 dollars each week. When will the two have the same amount of money?
17. Cell phone plan A charges \$75.00 each month and \$0.05 per text. Cell phone plan B charges \$109 dollars and \$0.00 per text.
    (a) At how many texts will the two plans charge the same?
    (b) Suppose you plan to text 3,000 times per month. Which plan should you choose? Why?
18. To rent a dunk tank Modern Rental charges \$150 per day. To rent the same tank, Budgetwise charges \$7.75 per hour.
    (a) When will the two companies charge the same?
    (b) You will need the tank for a 24-hour fund raise-a-thon. Which company should you choose?

CHAPTER 6 Linear Equations: Real World Applications

Chapter Outline

Algebra can be used to solve for the unknown quantity represented by a variable, but to solve real world applications, you need to be able to translate the situation into a mathematical equation. We have had some practice writing expressions from situations in Chapter 2. Now we will practice writing equations.

6.1 Writing Equations

Suppose there is a concession stand selling burgers and French fries. Each burger costs \$2.50 and each order of French fries costs \$1.75. You and your family will spend exactly \$25.00 on food. How many burgers can be purchased? How many orders of fries? How many of each type can be purchased if your family plans to buy a combination of burgers and fries?

The underlined word exactly lends a clue to the type of mathematical sentence you will need to write to model this situation.

These words can be used to symbolize the equal sign:

Exactly, equivalent, the same as, identical, is

The word *exactly* is synonymous with equal, so this word is directing us to write an equation. Using the methods previously learned, read every word in the sentence and translate each into mathematical symbols.

Example 1: Your family is planning to only purchase burgers. How many can be purchased with \$25.00?

Solution:

Step 1: Choose a variable to represent the unknown quantity, say b for burgers

Step 2: Write an equation to represent the situation: $2.50b = 25.00$

Step 3: Think. What number multiplied by 2.50 equals 25.00

The solution is 10, so your family can purchase exactly ten burgers.

Example 2: Translate the following into equations:

a) *9 less than twice a number is* 33.

b) *Five more than four times a number is* 21.

c) \$20.00 *was one-quarter of the money spent on pizza.*

Solutions:

a) Let a number be n. So, twice a number is $2n$.

Nine less than that is $2n - 9$.

The word *is* means the equal sign, so $2n - 9 = 33$

b) Let a number be x. So five more than four times a number is 21 can be written as: $4x+5=21$

c) Let of the money be m. The equation could be written as $\frac{1}{4}m=20.00$

Practice Set

Define the variables and translate the following statements into algebraic equations.

1. Peters Lawn Mowing Service charges \$10 per job and \$0.20 per square yard. Peter earns \$25 for a job.
2. Renting the ice-skating rink for a birthday party costs \$200 plus \$4 per person. The rental costs \$324 in total.
3. Renting a car costs \$55 per day plus \$0.45 per mile. The cost of the rental is \$100.
4. Nadia gave Peter 4 more blocks than he already had. He already had 7 blocks.
5. An amount of money is invested at 5% annual interest. The interest earned at the end of the year is equal to \$250.
6. You buy hamburgers at a fast food restaurant. A hamburger costs \$0.49. You have at \$3 to spend. Write an equation for the number of hamburgers you can buy.

6.2 Ratios and Proportions

There are many situations that can be represented with Algebraic equations. We will focus now on situations that involve proportions.

Ratios and **proportions** have a fundamental place in mathematics. They are used in geometry, size changes, and trigonometry. This lesson expands upon the idea of fractions to include ratios and proportions.

A **ratio** is a fraction comparing two things.

A **rate** is a fraction comparing two things with different units.

You have experienced rates many times: 65 $mi/hour$, \$1.99/$pound$, \$3.79/yd^2. You have also experienced ratios. A student to teacher ratio shows approximately how many students one teacher is responsible for in a school.

Example 1: *The State Dining Room in the White House measures approximately 48 feet long by 36 feet wide. Compare the length of the room to the width, and express your answer as a ratio.*

Solution:

$$\frac{48\ feet}{36\ feet} = \frac{4}{3}$$

The length of the State Dining Room is $\frac{4}{3}$ the width.

A **proportion** is a statement that two fractions are equal: $\frac{a}{b} = \frac{c}{d}$.

Example 2: *Is* $\frac{2}{3} = \frac{6}{12}$ *a proportion?*

Solution: Find the least common multiple of 3 and 12 to create a common denominator.

$$\frac{2}{3} = \frac{8}{12} \neq \frac{6}{12}$$

This is NOT a proportion because these two fractions are not equal.

A ratio can also be written using a colon instead of the fraction bar.

$\frac{a}{b} = \frac{c}{d}$ can also be read, a is to b as c is to d or $a : b = c : d$

You can use the Multiplication Property of Equality to find that:

If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$. This is true because we can multiply both sides of the equation by b which gives us $a = \frac{bc}{d}$, then we can multiply both sides of the equation by d, which gives us $ad = bc$.

ad is the product of the numerator of the first fraction and the denominator of the second fraction, and bc is the product of the denominator of the first fraction and the numerator of the second fraction. The are called **cross-products**.

Example 3: *Solve* $\frac{a}{9} = \frac{7}{6}$

Solution: Use the cross-products to solve.

$$6a = 7(9)$$
$$6a = 63$$

Solve for a:

$$a = 10.5$$

Consider the following situation: A train travels at a steady speed. It covers 15 miles in 20 minutes. How far will it travel in 7 hours, assuming it continues at the same rate? This is an example of a problem that can be solved using several methods, including proportions.

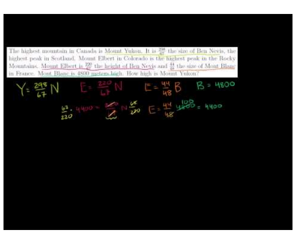
To solve using a proportion, you need to translate the statement into an algebraic sentence. The key to writing correct proportions is to keep the units the same in each fraction.

$$\frac{miles}{time} = \frac{miles}{time} \qquad \frac{miles}{time} \neq \frac{time}{miles}$$

You will be asked to solve this in the practice set.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. CK-12 Basic Algebra:Ratio and Proportion (10:25)



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Write the following comparisons as ratios. Simplify fractions where possible.

1. \$150 to \$3
2. 150 boys to 175 girls
3. 200 minutes to 1 hour
4. 10 days to 2 weeks

Write the following ratios as a unit rate.

5. 54 hotdogs to 12 minutes
6. 5000 lbs to 250 in^2
7. 20 computers to 80 students
8. 180 students to 6 teachers
9. 12 meters to 4 floors
10. 18 minutes to 15 appointments
11. Give an example of a proportion that uses the numbers 5, 1, 6, and 30
12. In the following proportion, identify the means and the extremes: $\frac{5}{12} = \frac{35}{84}$

Solve the following proportions.

13. $\frac{13}{6} = \frac{5}{x}$
14. $\frac{1.25}{7} = \frac{3.6}{x}$
15. $\frac{6}{19} = \frac{x}{11}$
16. $\frac{1}{x} = \frac{0.01}{5}$
17. $\frac{300}{4} = \frac{x}{99}$
18. $\frac{2.75}{9} = \frac{x}{\left(\frac{2}{9}\right)}$
19. $\frac{1.3}{4} = \frac{x}{1.3}$
20. $\frac{0.1}{1.01} = \frac{1.9}{x}$
21. $\frac{5p}{12} = \frac{3}{11}$
22. $-\frac{9}{x} = \frac{4}{11}$
23. $\frac{n+1}{11} = -2$
24. A restaurant serves 100 people per day and takes in \$908. If the restaurant were to serve 250 people per day, what might the cash collected be?
25. The highest mountain in Canada is Mount Yukon. It is $\frac{298}{67}$ the size of Ben Nevis, the highest peak in Scotland. Mount Elbert in Colorado is the highest peak in the Rocky Mountains. Mount Elbert is $\frac{220}{67}$ the height of Ben Nevis and $\frac{44}{48}$ the size of Mont Blanc in France. Mont Blanc is 4800 meters high. How high is Mount Yukon?
26. At a large high school it is estimated that two out of every three students have a cell phone, and one in five of all students have a cell phone that is one year old or less. Out of the students who own a cell phone, what proportion own a phone that is more than one year old?
27. The price of a Harry Potter Book on Amazon.com is \$10.00. The same book is also available used for \$6.50. Find two ways to compare these prices.
28. To prepare for school, you purchased 10 notebooks for \$8.79. How many notebooks can you buy for \$5.80?
29. It takes 1 cup mix and $\frac{3}{4}$ cup water to make 6 pancakes. How much water and mix is needed to make 21 pancakes?
30. Ammonia is a compound consisting of a 1:3 ratio of nitrogen and hydrogen atoms. If a sample contains 1,983 hydrogen atoms, how many nitrogen atoms are present?
31. The Eagles have won 5 out of their last 9 games. If this continues, how many games will they have won in the 63-game season?
32. Solve the train situation described earlier in this lesson.

6.3 Scale and Indirect Measurement

We are occasionally faced with having to make measurements of things that would be difficult to measure directly: the height of a tall tree, the width of a wide river, the height of the moons craters, even the distance between two cities separated by mountainous terrain. In such circumstances, measurements can be made **indirectly**, using proportions and similar triangles. Such indirect methods link measurement with geometry and numbers. In this lesson, we will examine some of the methods for making indirect measurements.

A map is a two-dimensional, geometrically accurate representation of a section of the Earths surface. Maps are used to show, pictorially, how various geographical features are arranged in a particular area. The **scale** of the map describes the relationship between distances on a map and the corresponding distances on the earth's surface. These measurements are expressed as a fraction or a ratio.

In the last lesson, you learned the different ways to write a fraction: using the fraction bar, a colon, and in words. Outside of mathematics books, ratios are often written as two numbers separated by a colon (:). Here is a table that compares ratios written in two different ways.

TABLE 6.1:

Ratio	Is Read As	Equivalent To
1:20	one to twenty	$\left(\frac{1}{20}\right)$
2:3	two to three	$\left(\frac{2}{3}\right)$
1:1000	one to one-thousand	$\left(\frac{1}{1000}\right)$

If a map had a scale of 1:1000 (one to one-thousand), one unit of measurement on the map (1 inch or 1 centimeter for example) would represent 1000 of the same units on the ground. A 1:1 (one to one) map would be a map as large as the area it shows!

Example : *Anne is visiting a friend in London, and is using the map above to navigate from Fleet Street to Borough Road. She is using a 1:100,000 scale map, where 1 cm* on the map represents *1 km in real life. Using a ruler, she measures the distance on the map as 8.8 cm. How far is the real distance from the start of her journey to the end?*

The scale is the ratio of distance on the map to the corresponding distance in real life and can be written as a proportion.

$$\frac{\text{dist.on map}}{\text{real dist.}} = \frac{1}{100,000}$$

By substituting known values, the proportion becomes:

$$\begin{aligned}
\frac{8.8\ cm}{\text{real dist.}(x)} &= \frac{1}{100,000} && \text{Cross multiply.}\\
880000\ cm &= x && 100\ cm = 1\ m.\\
x &= 8800\ m && 1000\ m = 1\ km.
\end{aligned}$$

The distance from Fleet Street to Borough Road is 8800 m or 8.8 km.

We could, in this case, use our intuition: the $1\ cm = 1\ km$ scale indicates that we could simply use our reading in centimeters to give us our reading in km. Not all maps have a scale this simple. In general, you will need to refer to the map scale to convert between measurements on the map and distances in real life!

Example 1: *Oscar is trying to make a scale drawing of the Titanic, which he knows was 883 feet long. He would like his drawing to be 1:500 scale. How long, in inches, must his sheet of paper be?*

Solution: We can reason that since the scale is 1:500 that the paper must be $\frac{883}{500} = 1.766\ feet$ long. Converting to inches gives the length at $12(1.766)\ in = 21.192\ in$.

The paper should be at least 22 *inches* long.

6.4 Percent Problems

Percent problems can be written as Algebraic equations. This section focuses on solving percept problems Algebraically. A **percent** is a ratio whose denominator is 100. Before we can use percents to solve problems, let's review how to convert percents to decimals and fractions and vice versa.

To convert a decimal to a percent, multiply the decimal by 100 and add a % sign.

Example 1: Convert 0.3786 to a percent.

$$0.3786 \times 100 = 37.86\%$$

To convert a percentage to a decimal, divide the percentage by 100 and drop the % sign.

Example 2: Convert 98.6% into a decimal.

$$98.6 \div 100 = 0.986$$

When converting fractions to percents, we can substitute $\frac{x}{100}$ for $x\%$, where x is the unknown.

Example 3: *Express* $\frac{3}{5}$ *as a percent.*

We start by representing the unknown as $x\%$ or $\frac{x}{100}$.

$$\begin{aligned}
\left(\frac{3}{5}\right) &= \frac{x}{100} && \text{Cross multiply.}\\
5x &= 100 \cdot 3\\
5x &= 300\\
x &= \frac{300}{5} = 60 && \text{Divide both sides by 5 to solve for } x.\\
\left(\frac{3}{5}\right) &= 60\%
\end{aligned}$$

Now that you remember how to convert between decimals and percents, you are ready for The Percent Equation.

The Percent Equation

The key words in a percent equation will help you translate it into a correct algebraic equation. Remember the equal sign symbolizes the word **is** and the multiplication symbol symbolizes the word **of.**

Example 4: *What is* 30% *of* 85.

Solution: First, translate into an equation.

$$n = 30\% \times 85$$

Convert the percent to a decimal and simplify.

$$n = 0.30 \times 85$$
$$n = 25.5$$

Example 5: *50 is 15% of what number?*

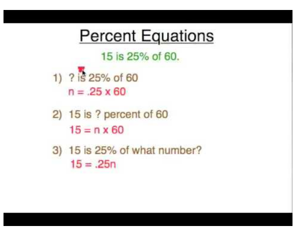
Solution: Translate into an equation.

$$50 = 15\% \times w$$

Rewrite the percent as a decimal and solve.

$$\begin{aligned}50 &= 0.15 \times w\\ \frac{50}{.15} &= \frac{0.15 \times w}{.15}\\ 333\frac{1}{3} &= w\end{aligned}$$

For more help with the percent equation, watch this 4-minute video recorded by Kens MathWorld. How to SolvePer centEquations (4:10)



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Finding the Percent of Change

A useful way to express changes in quantities is through percents. You have probably seen signs such as 20% more free, or save 35% today. When we use percents to represent a change, we generally use the formula:

$$\text{Percent change} = \left(\frac{\text{final amount - original amount}}{\text{original amount}}\right) \times 100\%$$

A **positive** percent change would thus be an **increase**, while a **negative** change would be a **decrease.**

Example 6: *A school of 500 students is expecting a 20% increase in students next year. How many students will the school have?*

Solution: Using the percent of change equation, translate the situation into an equation. Because the 20% is an increase, it is written as a positive value.

$$\text{Percent change} = \left(\frac{\text{final amount - original amount}}{\text{original amount}}\right) \times 100\%$$

$$\begin{aligned}
20\% &= \left(\frac{\text{final amount} - 500}{500}\right) \times 100\% && \text{Divide both sides by } 100\%. \\
& && \text{Let } x = \text{final amount.} \\
0.2 &= \frac{x-500}{500} && \text{Multiply both sides by 500.} \\
100 &= x - 500 && \text{Add 500 to both sides.} \\
600 &= x
\end{aligned}$$

The school will have 600 students next year.

Example 7: *A $150 mp3 player is on sale for 30% off. What is the price of the player?*

Solution: Using the percent of change equation, translate the situation into an equation. Because the 30% is a discount, it is written as a negative.

$$\text{Percent change} = \left(\frac{\text{final amount - original amount}}{\text{original amount}}\right) \times 100\%$$

$$\begin{aligned}
\left(\frac{x-150}{150}\right) \cdot 100\% &= -30\% && \text{Divide both sides by } 100\%. \\
\left(\frac{x-150}{150}\right) &= -0.3\% && \text{Multiply both sides by 150.} \\
x - 150 = 150(-0.3) &= -45 && \text{Add 150 to both sides.} \\
x &= -45 + 150 \\
x &= 105
\end{aligned}$$

The mp3 player will cost $105.

Many real situations involve percents. Consider the following.

Example 8

In 2004, the **US Department of Agriculture** had 112,071 employees, of which 87,846 were Caucasian. Of the remaining minorities, African-American and Hispanic employees had the two largest demographic groups, with 11,754 and 6,899 employees respectively.*

a) Calculate the total percentage of minority (non-Caucasian) employees at the USDA.

b) Calculate the percentage of African-American employees at the USDA.

c) Calculate the percentage of minority employees at the USDA who were neither African-American nor Hispanic.

Solution:

a) The ***total*** number of employees is 112,071. We know that the number of Caucasian employees is 87,846, which means that there must be $(112,071 - 87,846) = 24,225$ non-Caucasian employees.

$$\begin{aligned} \text{Rate} \times 112,071 &= 24,225 && \text{Divide both sides by } 112,071. \\ \text{Rate} &\approx 0.216 && \text{Multiply by 100 to obtain percent :} \\ \text{Rate} &\approx 21.6\% \end{aligned}$$

Approximately 21.6% of USDA employees in 2004 were from minority groups.

b) Total $= 112071$ Part $= 11754$

$$\begin{aligned} \text{Rate} \times 112071 &= 11754 && \text{Divide both sides by } 112071. \\ \text{Rate} &\approx 0.105 && \text{Multiply by 100 to obtain percent :} \\ \text{Rate} &\approx 10.5\% \end{aligned}$$

Approximately 10.5% of USDA employees in 2004 were African-American.

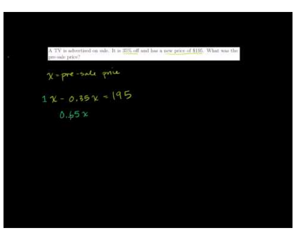
c) We now know there are 24225 non-Caucasian employees. That means there must be $(24225 - 11754 - 6899) = 5572$ minority employees who are neither African-American nor Hispanic. The part is 5572.

$$\begin{aligned} \text{Rate} \times 112071 &= 5572 && \text{Divide both sides by } 112071. \\ \text{Rate} &\approx 0.05 && \text{Multiply by 100 to obtain percent.} \\ \text{Rate} &\approx 5\% \end{aligned}$$

Approximately 5% of USDA minority employees in 2004 were neither African-American nor Hispanic.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. CK-12 Basic Algebra:PercentProblems (14:15)



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Express the following decimals as percents.

1. 0.011

2. 0.001
3. 0.91
4. 1.75
5. 20

Express the following fractions as a percent (round to two decimal places when necessary).

6. $\frac{1}{6}$
7. $\frac{5}{24}$
8. $\frac{6}{7}$
9. $\frac{11}{7}$
10. $\frac{13}{97}$

Express the following percentages as reduced fractions.

11. 11%
12. 65%
13. 16%
14. 12.5%
15. 87.5%

Find the following.

16. 32% of 600 is what number?
17. $\frac{3}{4}$% of 16 is what number?
18. 9.2% of 500 is what number
19. 8 is 20% of what number?
20. 99 is 180% of what number?
21. What percent of 7.2 is 45?
22. What percent of 150 is 5?
23. What percent of 50 is 2500?
24. A sweatshirt costs \$35. Find the total cost if the sales tax is 7.75%.
25. This year you got a 5% raise. If your new salary is \$45,000, what was your salary before the raise?
26. It costs \$250 to carpet a room that is $14\ ft \times 18\ ft$. How much does it cost to carpet a room that is $9\ ft \times 10\ ft$?
27. A department store has a 15% discount for employees. Suppose an employee has a coupon worth \$10 off any item and she wants to buy a \$65 purse. What is the final cost of the purse if the employee discount is applied before the coupon is subtracted?
28. A realtor earns 7.5% commission on the sale of a home. How much commission does the realtor make if the home sells for \$215,000?
29. The fire department hopes to raise \$30,000 to repair a fire house. So far the department has raised \$1,750.00. What percent is this of their goal?
30. A \$49.99 shirt goes on sale for \$29.99. By what percent was the shirt discounted?
31. A TV is advertised on sale. It is 35% off and has a new price of \$195. What was the pre-sale price?
32. An employee at a store is currently paid \$9.50 per hour. If she works a full year, she gets a 12% pay rise. What will her new hourly rate be after the raise?
33. Store A and Store B both sell bikes, and both buy bikes from the same supplier at the same prices. Store A has a 40% mark-up for their prices, while store B has a 90% mark-up. Store B has a permanent sale and will always sell at 60% off those prices. Which store offers the better deal?
34. 788 students were surveyed about their favorite type of television show. 18% stated that their favorite show was reality-based. How many students said their favorite show was reality-based?

CHAPTER 7

Literal Equations and Inequalities

Chapter Outline

7.1 Literal Equations

In the previous section, we looked at problem solving strategies using formulas. Formulas are examples of Literal Equations and in this section, we are going to look at Literal Equations in more depth.

Officially, a Literal Equation is an equation with several variables (more than one). Some of the more common Literal Equations that we see all the time are:

TABLE 7.1:

A = LW	Area = Length * Width	Area of a Rectangle
P = 2L + 2W	Perimeter = Twice the Length + Twice the Width	Perimiter of a Rectangle
A = πr^2	Area = Pi * the Square of the Radius	Area of a Circle
F = 1.8C + 32	Temperature in Fahrenheit = the temperature in Celsius * 1.8 + 32	Conversion from Celsius to Fahrenheit
D = RT	Distance = Rate * Time	Distance Formula
A = P + Prt	Accrued Value = Principal + Principal * Interest Rate * Time	Simple Interest Formula

Sometimes we need to rearrange a formula to solve for a different variable. For example, we may know the Area and the Length of a rectangle, and we want to find the width. Or we may know the time that weve been travelling and the total distance that weve travelled, and we want to find our average speed (rate). To do this, we need to isolate the variable of interest on one side of the equal sign, with all the other terms on the opposite side of the equal sign. Luckily, we can use the same techniques that we have been using to solve for the value of a variable.

Example 1: Find the Width of a Rectangle given the Area and the Length

Solution: The Area of a rectangle is given by the formula $A = lw$. We need to solve this equation for w.

$$A = lw \qquad \text{Area = Length * Width}$$

$$\frac{A}{l} = \frac{lw}{l} \qquad \text{Divide both sides by Length}$$

$$\frac{A}{l} = w \qquad \text{Equivalent form solved for Width}$$

Example 2: Find the Width of a Rectangle when given the Perimeter and Length

Solution: The perimeter of a rectangle is given by the formula $P = 2l + 2w$. We need to solve for w.

This one is actually a little more complicated. The formula has one term on the left hand side and two terms on the right. It's important to remember that we have to move one term to the left first.

$$\begin{aligned}
P &= 2l + 2w && \text{Perimeter = twice the Length + twice the Width} \\
P - 2l &= 2l + 2w - 2l && \text{Subtract 2l from both sides} \\
P - 2l &= 2w && \text{This leaves us with one term on the right} \\
\frac{P-2l}{2} &= \frac{2w}{2} && \text{Divide both sides by 2} \\
\frac{P-2l}{2} &= w && \text{Equivalent form solved for Width}
\end{aligned}$$

Example 3: Find the length of one side of a triangle, given the Perimeter and the other two sides.

Solution: The Perimeter of a triangle is the sum of the lengths of the three sides, so $P = a+b+c$. We need to solve for a, b or c.

This one seems to give people problems, so let's take a look at it.

$$\begin{aligned}
P &= a+b+c && \text{Perimeter = sum of the three sides. Let's solve for c} \\
P - a &= a+b+c-a && \text{Subtract a from both sides} \\
P - a &= b+c && \\
P - a - b &= b+c-b && \text{Subtract b from both sides} \\
P - a - b &= c && \text{Equivalent form solved for c}
\end{aligned}$$

Example 4: Find the voltage given the current and the power.

Solution: A formula that relates power with current and voltage is $I = \frac{P}{E}$, where I is the current, P is the power, and E is the voltage.

In this example, the variable you want to solve for is part of the denominator of a fraction.

$$\begin{aligned}
I &= \frac{P}{E} && \text{Amps = Watts/Voltage} \\
\left(\frac{E}{1}\right) I &= \frac{P}{E}\left(\frac{E}{1}\right) && \text{Multiply both sides E/1} \\
EI &= P && \text{Now E is in the Numerator on the left} \\
\frac{EI}{I} &= \frac{P}{I} && \text{Divide both sides by I} \\
E &= \frac{P}{I} && \text{Equivalent form solved for E}
\end{aligned}$$

Example 5: Problem Solving - Building a planter

Dave is building a planter for his wife. It's going to be two feet wide and four feet long. He wants the Planter to hold 20 cubic feet of soil to ensure the plants have room to grow. How high should the planter be?

Dave knows that the volume of a box (Rectangular Prism) is V = l*w*h so he decides to solve the equation for h and plug it into his calculator

$$V = lwh$$

$$\frac{V}{lw} = \frac{lwh}{lw} \qquad \text{Divide both sides by lw}$$

$$\frac{V}{lw} = h \qquad \text{Equivalent form solved for h}$$

Given the new formula, Dave plugs 20/(2*4) into his calculator and gets 2.5. Therefore, Dave should build his planter 2.5 feet high.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both.CK-12 Basic Algebra:Solvingfora variable (2:41)

For each of the following Equations, solve for the indicated variable

1. The formula for converting from Fahrenheit to Celsius is

Convert this formula so it solves for F

$$C = \frac{5}{9}(F - 32)$$

2. P = IRT solve for T

3. $C = 2\pi r$ solve for r

4. y = 5x - 6 solve for x

5. 4x - 3y = 6 solve for y

6. y = mx + b solve for b

7. ax + by = c solve for y

8. A = P + Prt solve for t

9. V = LWH solve for L

10. A = $4pr^2$ solve for r^2

11. V = pr^2h solve for h

12. 5x - y = 10 solve for x

13. A = (x + y)/2 solve for y

14. 12x 4y = 20 solve for y

15. A = 1/2h(b + c) solve for *b*

16. John knows that the formula to calculate how far he has traveled is average speed * time traveled. Write a literal equation that represents this formula using D for distance, r for average rate and t for time traveled. Then rewrite the formula, solving for average speed, r.

17. Mary paid $100 to set up a face painting booth at the local fair. She is going to charge customers $5 each. She figures out that her profits will be $5 per customer minus her $100 rental fee and comes up with the formula P = 5c - 100 where P is profit and *c* is the number of customers. She wants to earn at least $400 at the fair. Help Mary rewrite her formula so that she can calculate c. How many customers will she need?

7.2 Inequalities

Sometimes Things Are Not Equal

In some cases there are multiple answers to a problem or the situation requires something that is not exactly equal to another value. When a mathematical sentence involves something other than an equal sign, an **inequality** is formed.

Definition: An **algebraic inequality** is a mathematical sentence connecting an expression to a value, variable, or another expression with an inequality sign.

Listed below are the most common inequality signs.

$>$ greater than

$\geq$ greater than or equal to

$\leq$ less than or equal to

$<$ less than

$\neq$ not equal to

Below are several examples of inequalities.

$$3x < 5 \qquad \frac{3x}{4} \geq \frac{x}{2} - 3 \qquad 4 - x \leq 2x$$

Example 1: Translate the following into an inequality: Avocados cost \$1.59 per pound. How many pounds of avocados can be purchased for less than \$7.00?

Solution: Choose a variable to represent the number of pounds of avocados purchased, say a.

$$1.59(a) < 7$$

You will be asked to solve this inequality in the exercises

Checking the Solution to an Inequality

Unlike equations, inequalities typically have more than one solution. Checking solutions to inequalities are more complex than checking solutions to equations. The key to checking a solution to an inequality is to choose a number that occurs within the solution set.

Example 2: Check $m \leq 10$ is a solution to $4m+30 \leq 70$.

Solution: If the solution set is true, any value less than or equal to 10 should make the original inequality true.

Choose a value less than 10, say 4. Substitute this value for the variable m.

$$
\begin{aligned}
&4(4)+30\\
&16+30\\
&46 \leq 70
\end{aligned}
$$

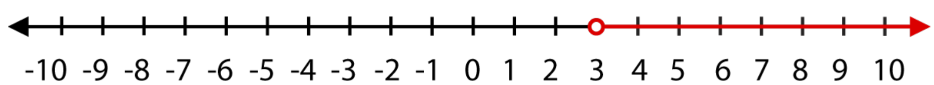
The value found when $m = 4$ is less than 70. Therefore, the solution set is true.

Why was the value 10 not chosen? Endpoints are not chosen when checking an inequality because the direction of the inequality needs to be tested. Special care needs to be taken when checking the solutions to an inequality.

Solutions to one-variable inequalities can be graphed on a number line or in a coordinate plane.

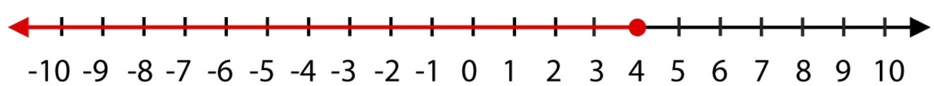
Example 3: *Graph the solutions to* $t > 3$ *on a number line.*

Solution: The inequality is asking for all real numbers 3 or larger.



You can also write inequalities given a number line of solutions.

Example 4: *Write the inequality pictured below.*



Solution: The value of four is colored in, meaning that four is a solution to the inequality. The red arrow indicates values less than four. Therefore, the inequality is:

$$x \leq 4$$

Inequalities that include the value are shown as $\leq$ *or* $\geq$. The line underneath the inequality stands for or equal to. We show this relationship by coloring in the circle above this value, as in the previous example. For inequalities without the or equal to the circle above the value remains unfilled.

Three Ways to Express Solutions to Inequalities

1. Inequality notation: The answer is expressed as an algebraic inequality, such as $d \leq \frac{1}{2}$.

2. Interval notation: This notation uses brackets to denote the range of values in an inequality.

- Square or closed brackets [ ] indicate that the number is **included** in the solution
- Round or open brackets ( ) indicate that the number is **not included** in the solution.

Interval notation also uses the concept of infinity ∞ and negative infinity $-\infty$. For example, for all values of d that are less than or equal to $\frac{1}{2}$, you could use set notation as follows: $\left(-\infty, \frac{1}{2}\right\}$

3. As a graphed sentence on a number line.

Example 5: (8, 24) states that the solution is all numbers between 8 and 24 but **does not include** the numbers 8 and 24.

[3, 12) states that the solution is all numbers between 3 and 12, **including** 3 but **not including** 12.

Inequalities Using Addition or Subtraction

To solve inequalities, you need some properties.

Addition Property of Inequality: For all real numbers a, b, and c:

If $x < a$, *then* $x+b < a+b$

If $x < a$, *then* $x-c < a-c$

The two properties above are also true for $\leq$ *or* $\geq$.

Because subtraction can also be thought of as **add the opposite,** these properties also work for subtraction situations.

Just like one-step equations, the goal is to **isolate the variable,** meaning to get the variable alone on one side of the inequality symbol. To do this, you will cancel the operations using inverses.

Example 6: *Solve for* $x: \ x-3 < 10$

Solution: To isolate the variable x, you must cancel subtract 3 using its inverse operation, addition.

$$\begin{aligned} x-3+3 &< 10+3 \\ x &< 13 \end{aligned}$$

Now, check your answer. Choose a number less than 13 and substitute it into your original inequality. If you choose 0, and substitute it you get:

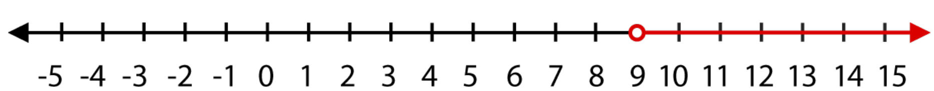
$$0-3 < 10 = -3 < 10$$

What happens at 13? What happens with numbers greater than 13?

Example 7: *Solve for* $x: \ x+4 > 13$

Solution:

To solve the inequality	$x+4 > 13$
Subtract 4 on both sides of the inequality.	$x+4-4 > 13-4$
Simplify.	$x > 9$



Writing Real Life Inequalities

As described in the chapter opener, inequalities appear frequently in real life. Solving inequalities is an important part of algebra.

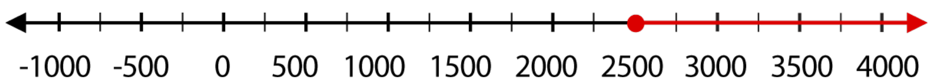
Example 8: Write the following statement as an algebraic inequality. You must maintain a balance of at least $2,500 in your checking account to avoid a finance charge.

Solution: The key phrase in this statement is at least. This means you can have $2,500 or more in your account to avoid a finance charge.

Choose the variable to describe the money in your account, say m.

Write the inequality $m \geq 2500$

Graph the solutions using a number line.

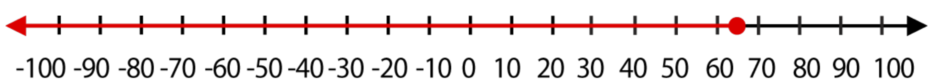


Example 9: *Translate into an algebraic inequality:* The speed limit is 65 miles per hour.

Solution: To avoid a ticket, you must drive 65 or less. Choose a variable to describe your possible speed, say s.

Write the inequality $s \leq 65$.

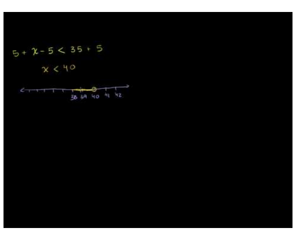
Graph the solutions to the inequality using a number line.



In theory, you cannot drive a negative number of miles per hour. This concept will be a focus later in this chapter.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. CK-12 Basic Algebra:Inequalities Using AdditionandSubtraction (7:48)



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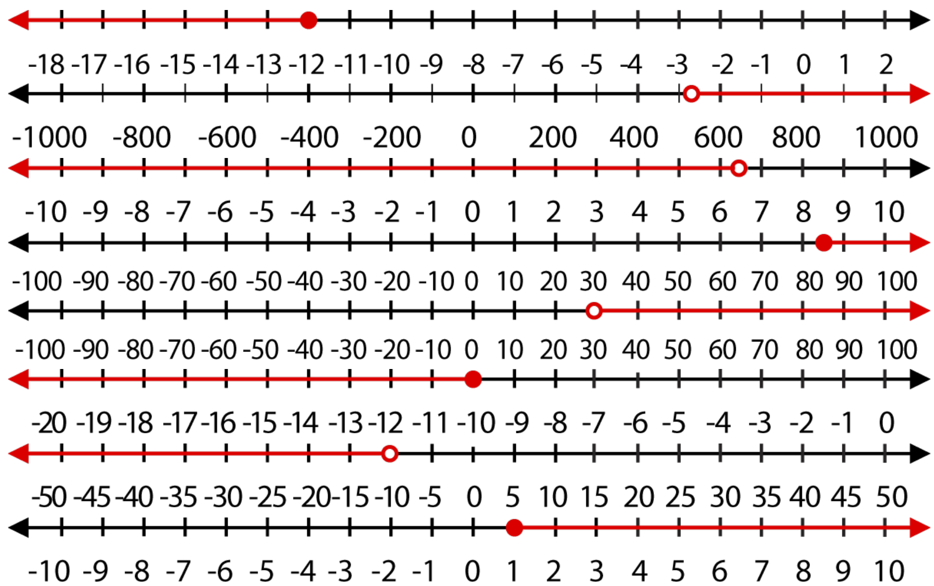
1. What are the three methods of writing the solution to an inequality?

Graph the solutions to the following inequalities using a number line.

2. $x < -3$

3. $x \geq 6$
4. $x > 0$
5. $x \leq 8$
6. $x < -35$
7. $x > -17$
8. $x \geq 20$
9. $x \leq 3$

Write the inequality that is represented by each graph.



10.
11.
12.
13.
14.
15.
16.
17.

Write the inequality given by the statement. Choose an appropriate letter to describe the unknown quantity.

18. You must be at least 48 inches tall to ride the Thunderbolt Rollercoaster.
19. You must be younger than 3 years old to get free admission at the San Diego Zoo.
20. Charlie needs more than $1,800 to purchase a car.
21. Cheryl can have no more than six pets at her house.
22. The shelter can house no more than 16 rabbits.

Solve each inequality and graph the solution on the number line.

23. $x - 1 > -10$
24. $x - 1 \leq -5$
25. $-20 + a \geq 14$
26. $x + 2 < 7$
27. $x + 8 \leq -7$
28. $5 + t \geq \frac{3}{4}$
29. $x - 5 < 35$
30. $15 + g \geq -60$
31. $x - 2 \leq 1$
32. $x - 8 > -20$
33. $11 + q > 13$
34. $x + 65 < 100$
35. $x - 32 \leq 0$
36. $x + 68 \geq 75$
37. $16 + y \leq 0$

7.3 Inequalities Using Multiplication and Division

Equations are mathematical sentences in which the two sides have the same weight. By adding, subtracting, multiplying or dividing the same value to both sides of the equation, the balance stays in check. However, inequalities begin off-balance. When you perform inverse operations, the inequality will remain off-balance. This is true with inequalities involving both multiplication and division.

Before we can begin to solve inequalities involving multiplication or division, you need to know two properties, the Multiplication Property of Inequalities and the Division Property of Inequalities.

Multiplication Property of Inequality: For all real positive numbers a, b, and c:

If $x < a$, *then* $x(c) < a(c)$.

If $x > a$, *then* $x(c) > a(c)$.

Division Property of Inequality: For all real positive numbers a, b, and c:

If $x < a$, *then* $x \div (c) < a \div (c)$.

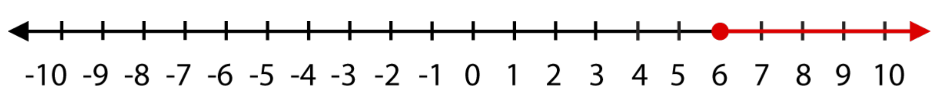
If $x > a$, *then* $x \div (c) > a \div (c)$.

Consider the inequality $2x \geq 12$. To find the solutions to this inequality, we must isolate the variable x by using the inverse operation of multiply by 2, which is dividing by 2.

$$\begin{aligned} 2x &\geq 12 \\ \frac{2x}{2} &\geq \frac{12}{2} \\ x &\geq 6 \end{aligned}$$

This solution can be expressed in four ways. One way is already written, $x \geq 6$. Below are the three remaining ways to express this solution:

- $\{x | x \geq 6\}$
- $[6, \infty)$
- Using a number line:



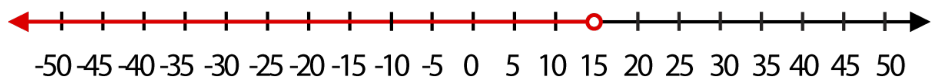
Example 1: *Solve for* $y : \frac{y}{5} \leq 3$. *Express the solution using all four methods.*

Solution: The inequality above is read, y divided by 5 is less than or equal to 3. To isolate the variable y, you must cancel division using its inverse operation, multiplication.

$$\begin{aligned} \frac{y}{5} \cdot \frac{5}{1} &\leq 3 \cdot \frac{5}{1} \\ y &\leq 15 \end{aligned}$$

One method of writing the solution is $y \leq 15$.

The other three are:

- $(-\infty, 15]$
- $\{y|y \leq 15\}$
- 

Multiplying and Dividing an Inequality by a Negative Number

Notice the two properties in this lesson focused on c being only positive. This is because those particular properties of multiplication and division do not apply when the number being multiplied (or divided) is negative.

Think of it this way. When you multiply a value by -1, the number you get is the negative of the original.

$$6(-1) = -6$$

Multiplying each side of a sentence by -1 results in the opposite of both values.

$$\begin{aligned}5x(-1) &= 4(-1)\\ -5x &= -4\end{aligned}$$

When multiplying by a negative, you are doing the opposite of everything in the sentence, including the verb.

$$\begin{aligned}x &> 4\\ x(-1) &> 4(-1)\\ -x &< -4\end{aligned}$$

This concept is summarized below.

Multiplication/Division Rule of Inequality: For any real number a, and any **negative** number c,

If $x < a$, *then* $x \cdot c > a \cdot c$

If $x < a$, *then* $\frac{x}{c} > \frac{a}{c}$

As with the other properties of inequalities, these also hold true for $\leq$ *or* $\geq$.

Example 2: *Solve for* $r : -3r < 9$

Solution: To isolate the variable r, we must cancel multiply by -3 using its inverse operation, dividing by -3.

$$\frac{-3r}{-3} < \frac{9}{-3}$$

Since you are dividing by -3, everything becomes opposite, including the inequality sign.

$$r > -3$$

Example 3: *Solve for* $p : 12p < -30$

Solution: To isolate the variable p, we must cancel multiply by 12 using its inverse operation, dividing by 12.

$$\frac{12p}{12} < \frac{-30}{12}$$

Because 12 is **not** negative, you do **not** switch the inequality sign.

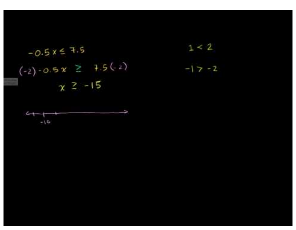
$$p < \frac{-5}{2}$$

In set notation, the solution would be: $\left(-\infty, \frac{-5}{2}\right)$

Multimedia Link: For more help with solving inequalities involving multiplication and division, visit Khan Academy's website. http://khanexercises.appspot.com/video?v=PNXozoJWsWc

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. CK-12 Basic Algebra:Inequalities Using Multiplication and Division (10:27)



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1. In which cases do you change the inequality sign?

Solve each inequality. Give the solution using inequality notation and with a solution graph.

2. $3x \leq 6$
3. $\frac{x}{5} > -\frac{3}{10}$
4. $-10x > 250$
5. $\frac{x}{-7} \geq -5$
6. $9x > -\frac{3}{4}$
7. $\frac{x}{-15} \leq 5$
8. $620x > 2400$

9. $\frac{x}{20} \geq -\frac{7}{40}$
10. $-0.5x \leq 7.5$
11. $75x \geq 125$
12. $\frac{x}{-3} > -\frac{10}{9}$
13. $\frac{k}{-14} \leq 1$
14. $\frac{x}{-15} < 8$
15. $\frac{x}{2} > 40$
16. $\frac{x}{-3} \leq -12$
17. $\frac{x}{25} < \frac{3}{2}$
18. $\frac{x}{-7} \geq 9$
19. $4x < 24$
20. $238 < 14d$
21. $-19m \leq -285$
22. $-9x \geq -\frac{3}{5}$
23. $-5x \leq 21$
24. The width of a rectangle is 16 inches. Its area is greater than 180 square inches.

- - Write an inequality to represent this situation.
  - Graph the possible lengths of the rectangle.

24. Ninety percent of some number is at most 45.

- - Write an inequality to represent the situation.
  - Write the solutions as an algebraic sentence.

25. Doubling Marthas jam recipe yields at least 22 pints.

- - Write an inequality to represent the situation.
  - Write the solutions using interval notation.

7.4 Multi-Step Inequalities

The previous two lessons focused on one-step inequalities. Inequalities, like equations, can require several steps in order to isolate the variable. These inequalities are called **multi-step inequalities.** With the exception of the Multiplication/Division Rule of Inequality, the process of solving multi-step inequalities is identical to solving multi-step equations.

Procedure to Solve an Inequality:

1. Remove any parentheses by using the Distributive Property.
2. Simplify each side of the inequality by combining like terms.
3. Isolate the ax term. Use the Addition Property of Inequality to get the variable on one side of the equal sign and the numerical values on the other.
4. Isolate the variable. Use the Multiplication Property of Inequality to get the variable alone on one side of the equation.
   (a) Remember to reverse the inequality sign if you are multiplying or dividing by a negative.
5. Check your solution.

Example 1: *Solve for* $w: \ 6x-5<10$

Solution: Begin by using the checklist above.

1. Parentheses? No

2. Like terms on the same side of inequality? No

3. Isolate the ax term using the Addition Property.

$$6x-5+5<10+5$$

Simplify:

$$6x<15$$

4. Isolate the variable using the Multiplication or Division Property.

$$\frac{6x}{6}<\frac{15}{6}=x<\frac{5}{2}$$

5. Check your solution. Choose a number less than 2.5, say 0 and check using the original inequality.

$$\begin{aligned}6(0)-5&<10?\\-5&<10\end{aligned}$$

Yes, the answer checks. $x<2.5$

Example 2: *Solve for* $x: \ -9x < -5x - 15$

Solution: Begin by using the checklist above.

1. Parentheses? No

2. Like terms on the same side of inequality? No

3. Isolate the *ax* term using the Addition Property.

$$-9x + 5x < -5x + 5x - 15$$

Simplify:

$$-4x < -15$$

4. Isolate the variable using the Multiplication or Division Property.

$$\frac{-4x}{-4} < \frac{-15}{-4}$$

Because the number you are dividing by is negative, you must reverse the inequality sign.

$$x > \frac{15}{4} \rightarrow x > 3\frac{3}{4}$$

5. Check your solution by choosing a number larger than 3.75, say 10.

$$\begin{aligned} &-9(10) < -5(10) - 15? \\ &\checkmark \ -90 < -65 \end{aligned}$$

Example 3: *Solve for* $x: \ 4x - 2(3x - 9) \leq -4(2x - 9)$

Solution: Begin by using the previous checklist.

1. Parentheses? Yes. Use the Distributive Property to clear the parentheses.

$$4x + (-2)(3x) + (-2)(-9) \leq -4(2x) + (-4)(-9)$$

Simplify:

$$4x - 6x + 18 \leq -8x + 36$$

2. Like terms on the same side of inequality? Yes. Combine these

$$-2x + 18 \leq -8x + 36$$

3. Isolate the ax term using the Addition Property.

$$-2x+8x+18 \leq -8x+8x+36$$

Simplify:

$$6x+18 \leq 36$$
$$6x+18-18 \leq 36-18$$

4. Isolate the variable using the Multiplication or Division Property.

$$\frac{6x}{6} \leq \frac{18}{6} \rightarrow x \leq 3$$

5. Check your solution by choosing a number less than 3, say -5.

$$4(-5)-2(3 \cdot -5-9) \leq -4(2 \cdot -5-9)$$
$$\checkmark\ 28 < 76$$

Identifying the Number of Solutions to an Inequality

Inequalities can have infinitely many solutions, no solutions, or a finite set of solutions. Most of the inequalities you have solved to this point have an infinite amount of solutions. By solving inequalities and using the context of a problem, you can determine the number of solutions an inequality may have.

Example 4: Find the solutions to $x-5 > x+6$

Solution: Begin by isolating the variable using the Addition Property of Inequality.

$$x-x-5 > x-x+6$$

Simplify.

$$-5 > 6$$

This is an untrue inequality. Negative five is never greater than six. Therefore, the inequality $x-5 > x+6$ has no solutions.

Previously we looked at the following sentence. The speed limit is 65 miles per hour.

The algebraic sentence is: $s \leq 65$.

Example 5: Find the solutions to $s \leq 65$.

Solution: The speed at which you drive cannot be negative. Therefore, the set of possibilities using interval notation is [0, 65].

Solving Real-World Inequalities

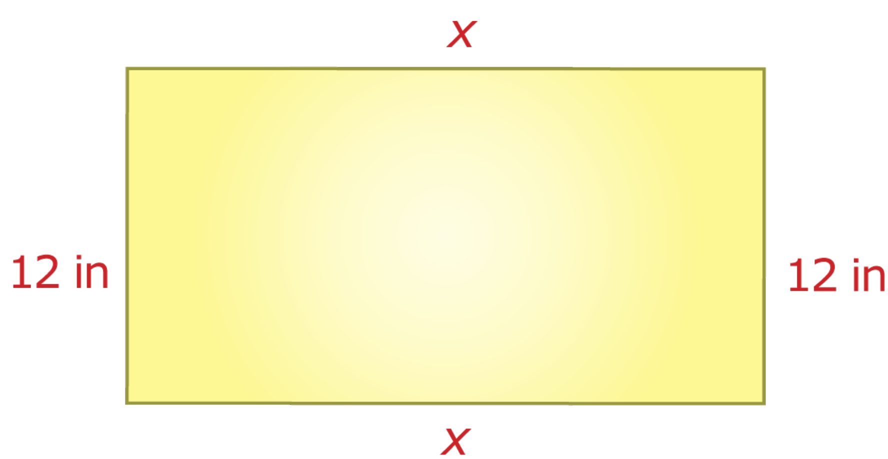
Example 6: In order to get a bonus this month, Leon must sell at least 120 newspaper subscriptions. He sold 85 subscriptions in the first three weeks of the month. How many subscriptions must Leon sell in the last week of the month?

Solution: The amount of subscriptions Leon needs is at least 120. Choose a variable to represent the varying quantitythe number of subscriptions, say n. The inequality that represents the situation is $n+85 \geq 120$.

Solve by isolating the variable n. $n \geq 35$

Leon must sell 35 or more subscriptions in order to receive his bonus.

Example 7: The width of a rectangle is 12 inches. What must the length be if the perimeter is at least 180 inches? Diagram not drawn to scale.



Solution: The perimeter is the sum of all the sides.

$$12+12+x+x \geq 180$$

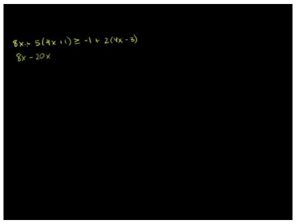
Simplify and solve for the variable x:

$$\begin{aligned} 12+12+x+x \geq 180 &\rightarrow 24+2x \geq 180 \\ 2x &\geq 156 \\ x &\geq 78 \end{aligned}$$

The length of the rectangle must be 78 inches or larger.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. CK-12 Basic Algebra:Multi-Step Inequalities (8:02)



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Solve each of the following inequalities and graph the solution set.

1. $6x - 5 < 10$
2. $-9x < -5x - 15$
3. $-\frac{9x}{5} \leq 24$
4. $\frac{9x}{5} - 7 \geq -3x + 12$
5. $\frac{5x-1}{4} > -2(x+5)$
6. $4x + 3 < -1$
7. $2x < 7x - 36$
8. $5x > 8x + 27$
9. $5 - x < 9 + x$
10. $4 - 6x \leq 2(2x+3)$
11. $5(4x+3) \geq 9(x-2) - x$
12. $2(2x-1) + 3 < 5(x+3) - 2x$
13. $8x - 5(4x+1) \geq -1 + 2(4x-3)$
14. $2(7x-2) - 3(x+2) < 4x - (3x+4)$
15. $\frac{2}{3}x - \frac{1}{2}(4x-1) \geq x + 2(x-3)$
16. At the San Diego Zoo, you can either pay $22.75 for the entrance fee or $71 for the yearly pass which entitles you to unlimited admission. At most how many times can you enter the zoo for the $22.75 entrance fee before spending more than the cost of a yearly membership?
17. Proteeks scores for four tests were 82, 95, 86 and 88. What will he have to score on his last test to average at least 90 for the term?
18. Raul is buying ties and he wants to spend $200 or less on his purchase. The ties he likes the best cost $50. How many ties could he purchase?
19. Virena's Scout Troup is trying to raise at least $650 this spring. How many boxes of cookies must they sell at $4.50 per box in order to reach their goal?

Chapter 8 Graphs and Graphing Linear Equations

Chapter Outline

The ability to graph linear equations is important in mathematics. In fact, graphing equations and solving equations are two of the most important concepts in mathematics. If you master these, all mathematical subjects will be much easier, even Calculus!

This chapter focuses on the visual representations of data. You will learn how graphs are created using the Coordinate Plane and how to create and interpret graphs using sets of data. Finally, you will learn what defines a good graph and what mistakes to avoid when creating a graph.

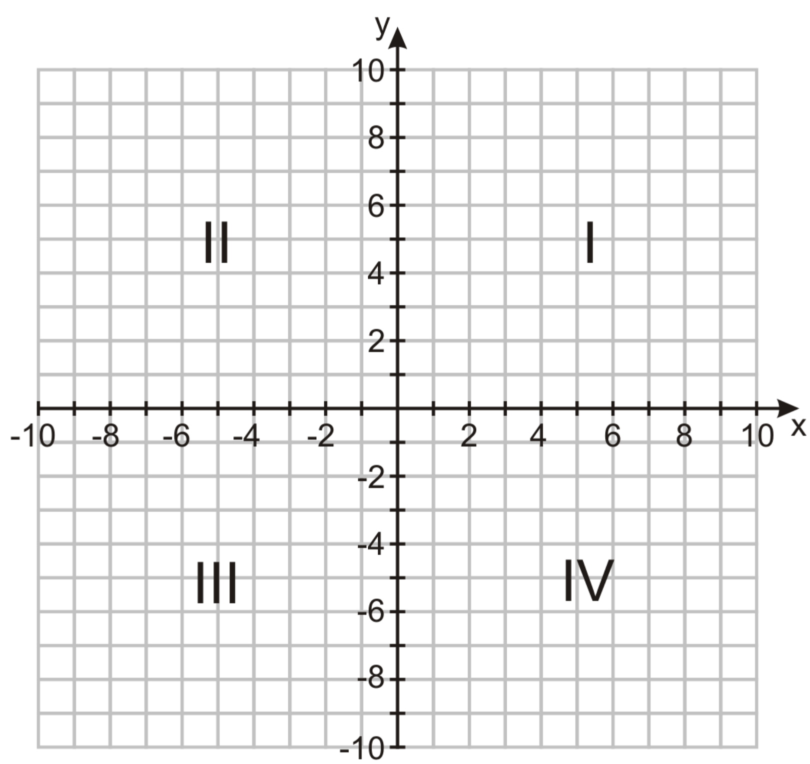
8.1 The Coordinate Plane

Introduction

Lydia lives 2 blocks north and one block east of school; Travis lives three blocks south and two blocks west of school. What's the shortest line connecting their houses?

The Coordinate Plane

We've seen how to represent numbers using number lines; now we'll see how to represent sets of numbers using a **coordinate plane.** The coordinate plane can be thought of as two number lines that meet at right angles. The horizontal line is called the $x-$**axis** and the vertical line is the $y-$**axis**. Together the lines are called the **axes**, and the point at which they cross is called the **origin**. The axes split the coordinate plane into four **quadrants**, which are numbered sequentially (I, II, III, IV) moving counter-clockwise from the upper right.



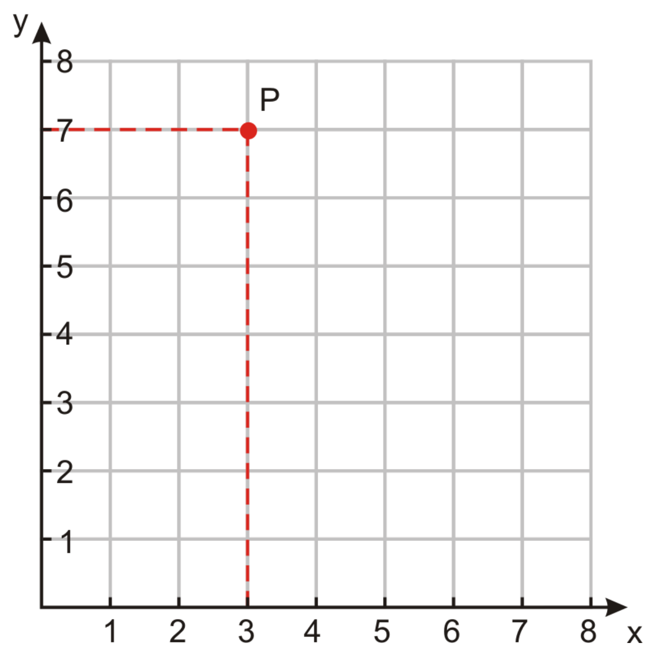
Identify Coordinates of Points

When given a point on a coordinate plane, it's easy to determine its **coordinates**. The coordinates of a point are two numbers - written together they are called an **ordered pair**. The numbers describe how far along the $x-$axis and $y-$axis the point is. The ordered pair is written in parentheses, with the $x-$**coordinate** (also called the **abscissa**) first and the $y-$**coordinate** (or the **ordinate**) second.

$(1,7)$	An ordered pair with an $x-$ value of one and a $y-$ value of seven
$(0,5)$	An ordered pair with an $x-$ value of zero and a $y-$ value of five
$(-2.5,4)$	An ordered pair with an $x-$ value of -2.5 and a $y-$ value of four
$(-107.2,-0.005)$	An ordered pair with an $x-$ value of -107.2 and a $y-$ value of -0.005

Identifying coordinates is just like reading points on a number line, except that now the points do not actually lie **on** the number line! Look at the following example.

Example 1



Find the coordinates of the point labeled P in the diagram above

Solution

Imagine you are standing at the origin (the point where the $x-$axis meets the $y-$axis). In order to move to a position where P was directly above you, you would move 3 units to the **right** (we say this is in the **positive** $x-$direction).

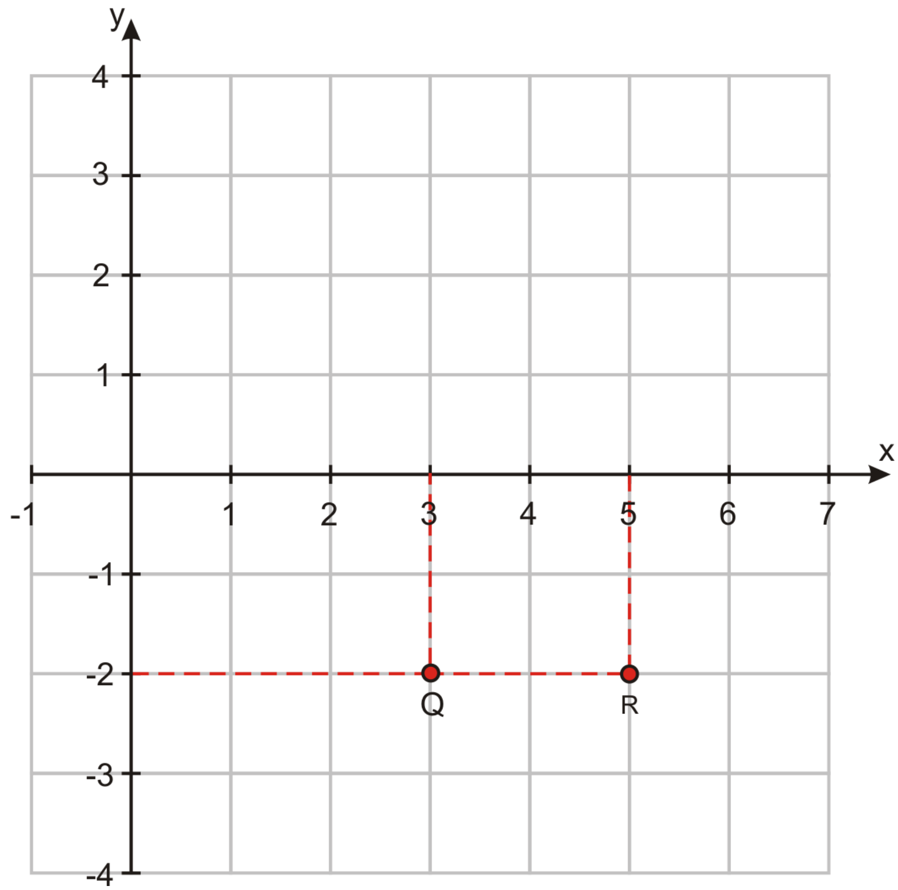
The $x-$coordinate of P is +3.

Now if you were standing at the 3 marker on the $x-$axis, point P would be 7 units **above** you (above the axis means it is in the **positive** y direction).

The $y-$coordinate of P is +7.

The coordinates of point P are (3, 7).

Example 2



Find the coordinates of the points labeled Q and R in the diagram to the right.

Solution

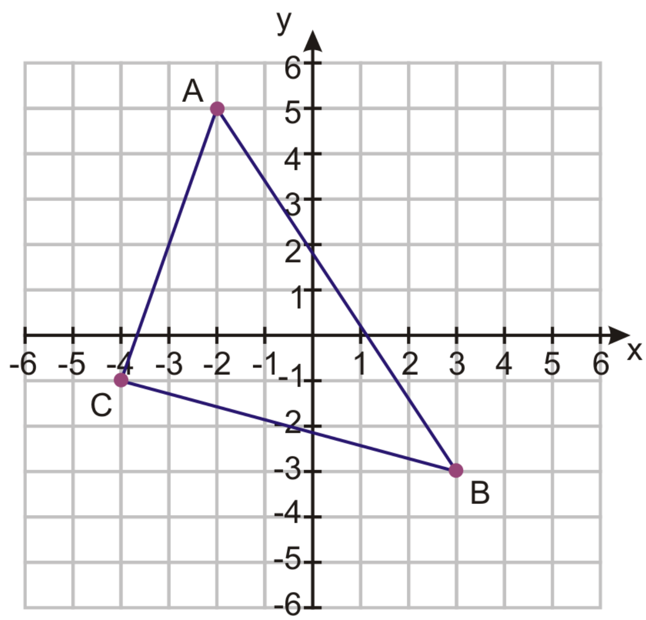
In order to get to Q we move three units to the right, in the positive $x-$direction, then two units **down**. This time we are moving in the **negative** $y-$direction. The $x-$coordinate of Q is +3, the $y-$coordinate of Q is -2.

The coordinates of R are found in a similar way. The $x-$coordinate is +5 (five units in the positive $x-$direction) and the $y-$coordinate is again -2.

The coordinates of Q are (3, -2). The coordinates of R are (5, -2).

Example 3

Triangle ABC is shown in the diagram to the right. Find the coordinates of the vertices A, B and C.



Point A:

$x-\text{coordinate} = -2$

$y-\text{coordinate} = +5$

Point B:

$x-\text{coordinate} = +3$

$y-\text{coordinate} = -3$

Point C:

$x-\text{coordinate} = -4$

$y-\text{coordinate} = -1$

Solution

$A(-2,5)$

$B(3,-3)$

$C(-4,-1)$

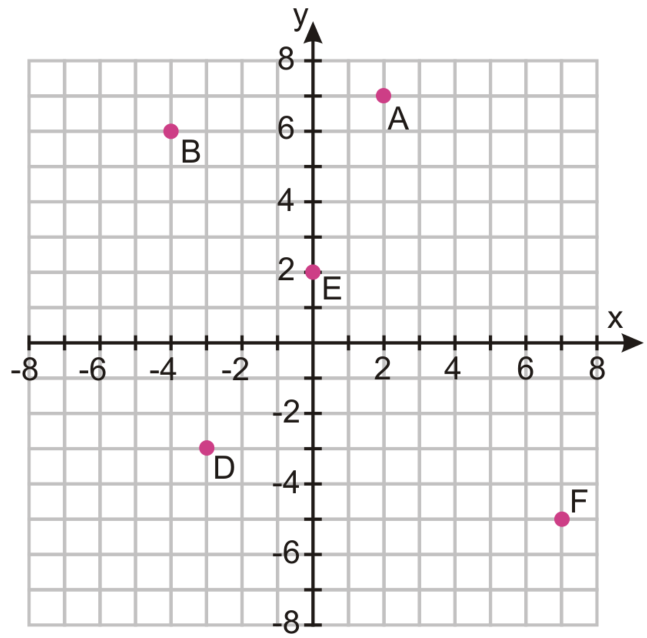
Plot Points in a Coordinate Plane

Plotting points is simple, once you understand how to read coordinates and read the scale on a graph. As a note on scale, in the next two examples pay close attention to the labels on the axes.

Example 4

Plot the following points on the coordinate plane.

$A(2,7) \quad B(-4,6) \quad D(-3,-3) \quad E(0,2) \quad F(7,-5)$



Point $A(2,7)$ is 2 units right, 7 units up. It is in Quadrant I.

Point $B(-4,6)$ is 4 units left, 6 units up. It is in Quadrant II.

Point $D(-3,-3)$ is 3 units left, 3 units down. It is in Quadrant III.

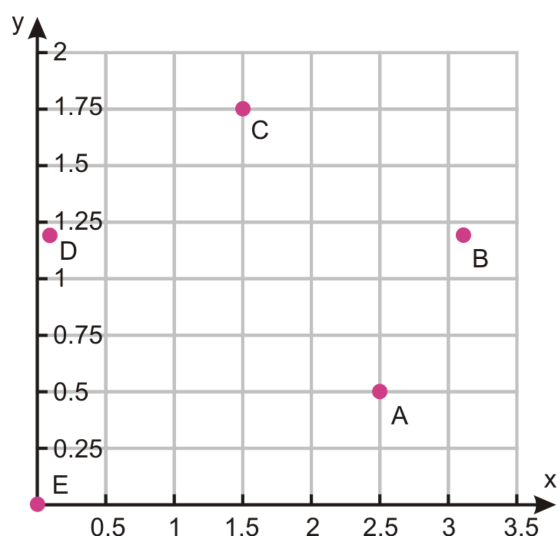
Point $E(0,2)$ is 2 units up from the origin. It is right on the $y-$axis, between Quadrants I and II.

Point $F(7,-5)$ is 7 units right, 5 units down. It is in Quadrant IV.

Example 5

Plot the following points on the coordinate plane.

$A(2.5, 0.5)$ $B(\pi, 1.2)$ $C(2, 1.75)$ $D(0.1, 1.2)$ $E(0,0)$



Here we see the importance of choosing the right scale and range for the graph. In Example 4, our points were scattered throughout the four quadrants. In this case, all the coordinates are positive, so we don't need to show the negative values of x or y. Also, there are no $x-$values bigger than about 3.14, and 1.75 is the largest value of y. We can therefore show just the part of the coordinate plane where $0 \le x \le 3.5$ and $0 \le y \le 2$.

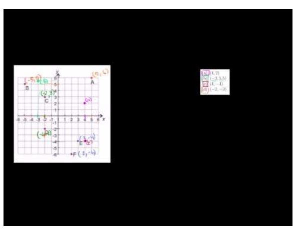
Here are some other important things to notice about this graph:

- The tick marks on the axes don't correspond to unit increments (i.e. the numbers do not go up by one each time). This is so that we can plot the points more precisely.
- The scale on the $x-$axis is different than the scale on the $y-$axis, so distances that look the same on both axes are actually greater in the $x-$direction. Stretching or shrinking the scale in one direction can be useful when the points we want to plot are farther apart in one direction than the other.
- The tick marks are equally spaced on each axis. The axes are like number lines - it is important that each tick mark represents the same increment.

For more practice locating and naming points on the coordinate plane, try playing the Coordinate Plane Game at **http://tinyurl.com/72myodv.**

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. CK-12 Basic Algebra:TheCoordinate Plane (6:50)

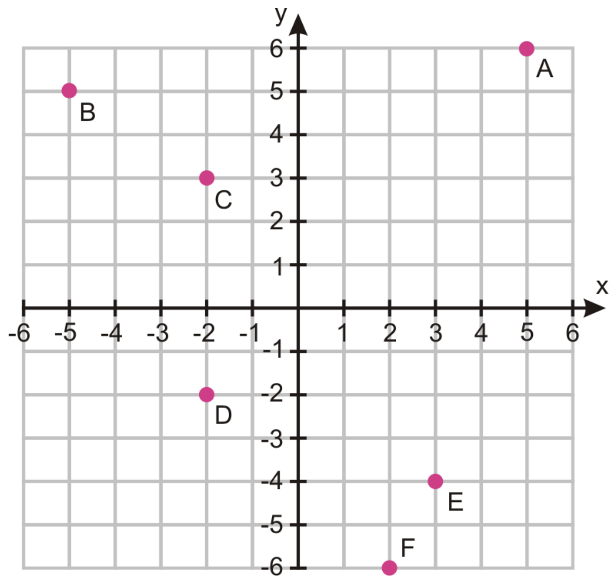


MEDIA

Click image to the left for more content.

In questions 1-6, identify the coordinate of the given letter.

1. D
2. A
3. F
4. E
5. B
6. C



Graph the following ordered pairs on one Cartesian plane. Identify the quadrant where each ordered pair is located.

7. (4, 2)
8. (-3, 5.5)
9. (4, -4)
10. (-2, -3)
11. $\left(\frac{1}{2}, -\frac{3}{4}\right)$
12. (-0.75, 1)
13. $\left(-2\frac{1}{2}, -6\right)$
14. (1.60, 4.25)

Using the directions given in each problem, find and graph the coordinates on a Cartesian plane.

15. Six left, four down
16. One-half right, one-half up
17. Three right, five down
18. Nine left, seven up
19. Four and one-half left, three up
20. Eight right, two up
21. One left, one down
22. One right, three-quarter down
23. Plot the vertices of triangle $ABC : (0,0), (4,-3), (6,2)$
24. The following three points are three vertices of square $ABCD$. Plot them on a coordinate plane then determine what the coordinates of the fourth point, D, would be. Plot that point and label it. $A(-4,-4)$ $B(3,-4)$ $C(3,3)$
25. Does the ordered pair (2, 0) lie in a quadrant? Explain your thinking.
26. Why do you think (0, 0) is called the origin?
27. Ian has the following collection of data. Graph the ordered pairs and make a conclusion from the graph.

TABLE 8.1:

Year	% of Men Employed in the United States
1973	75.5
1980	72.0
1986	71.0

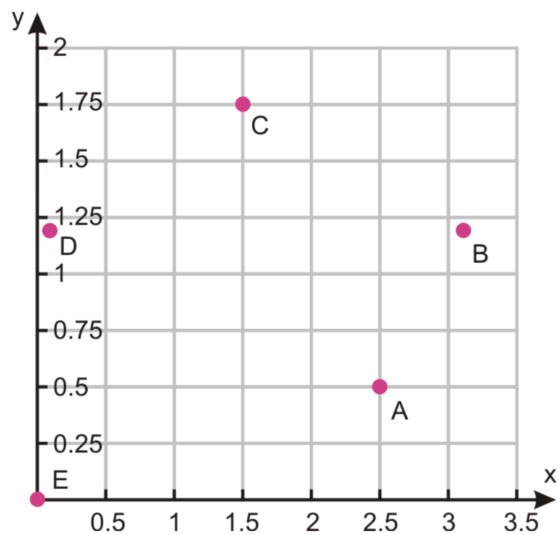
TABLE 8.1: (continued)

Year	% of Men Employed in the United States
1992	69.8
1997	71.3
2002	69.7
2005	69.6
2007	69.8
2009	64.5

Words of Wisdom from the Graphing Plane

Not all axes will be labeled for you. There will be many times you are required to label your own axes. Some problems may require you to graph only the first quadrant. Others need two or all four quadrants. The tic marks do not always count by ones. They can be marked in increments of 2, 5, or even $\frac{1}{2}$. The axes do not even need to have the same increments! The Cartesian plane below shows an example of this.

The increments by which you count your axes should MAXIMIZE the clarity of the graph.



8.2 What makes a good graph

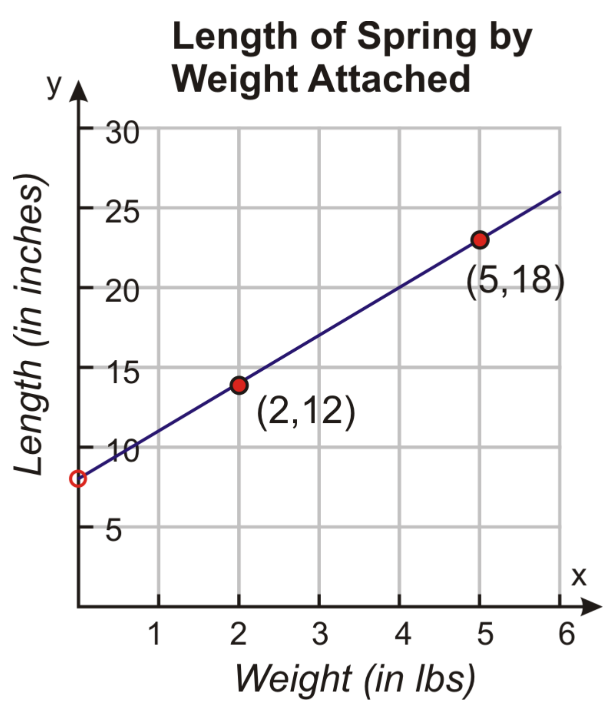
When creating a graph, there are guidelines to follow to ensure that it is easy to read and easy to interpret.

1. The horizontal axis should be properly labeled with the name and units of the input variable.2. The vertical axis should be properly labeled with the name and units of the output variable.3. Use an appropriate scale.

- Start at or just below the lowest replacement value.
- End at or just above the highest replacement value.
- Scale the graph so the adjacent tick marks are equal distance apart.
- Use numbers that make sense for the given data set.
- The axes meet at (0,0) Use a "//" between the origin and the first tick mark if the scale does not begin at 0.

Example 1

The graph shown in Example 3 is a very good example of how a graph should be created. It has all the elements of a good graph as noted above.



1. Both the horizontal axis and vertical axis are labeled with the name of the variable and the unit. The horizontal axis has the variable name Weight and the unit Pounds (lbs) The vertical axis is labeled with the variable name Length and the unit Inches. If the units were not included, there would be no way to know what unit of weight or length was being used
2. The horizontal and vertical scales both start with 0 and are consistent. Note that they are not the same scale. Weight changes by a scale of 1 and Length changes by a scale of 5.
3. The numbers for both scales make sense for the data and make good use of the space available.
4. The data points are labeled with the x and y value

Example 2

The last example that we are going to look at in this lesson is an example of a "bad" graph. It has examples of everything you should "not" do when creating a graph.

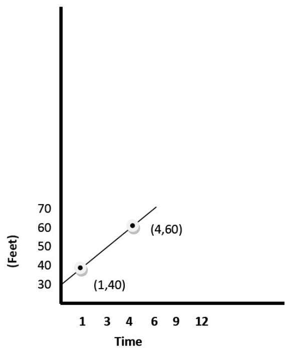


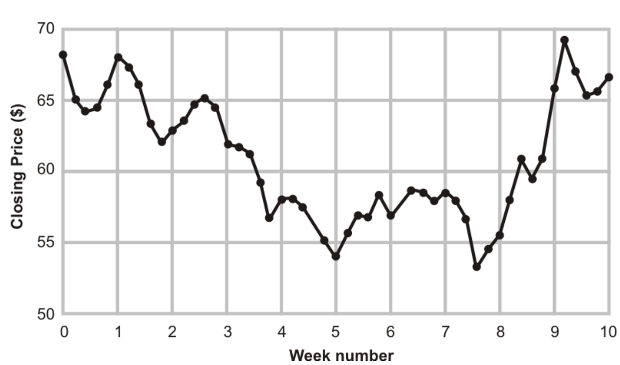
FIGURE 8.1

1. The Horizontal Axis does not have the units. It is impossible to tell if Time is being measured in Minutes, Seconds or something else.
2. The Vertical Axis does not have the Variable Name. What is being measured in Feet?
3. The horizontal scale is not consistent and the vertical scale does not start at 0
4. The graph makes very poor use of the available space

Remember as you create your graphs that labeling is critical. It is important that the graph be both readable and accurate. Follow the guidelines above at all times to ensure good practice.

Analyzing Graphs

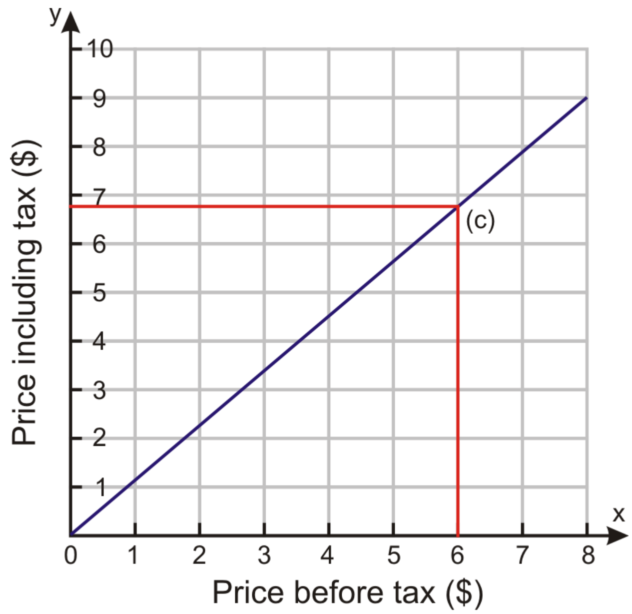
We often use graphs to represent relationships between two linked quantities. It's useful to be able to interpret the information that graphs convey. For example, the chart below shows a fluctuating stock price over ten weeks. You can read that the index closed the first week at about \$68, and at the end of the third week it was at about \$62. You may also see that in the first five weeks it lost about 20% of its value, and that it made about 20% gain between weeks seven and ten. Notice that this relationship is discrete, although the dots are connected to make the graph easier to interpret.



Analyzing graphs is a part of life - whether you are trying to decide to buy stock, figure out if your blog readership is increasing, or predict the temperature from a weather report. Many graphs are very complicated, so for now we'll start off with some simple linear conversion graphs. Algebra starts with basic relationships and builds to more complicated tasks, like reading the graph above.

Example 3

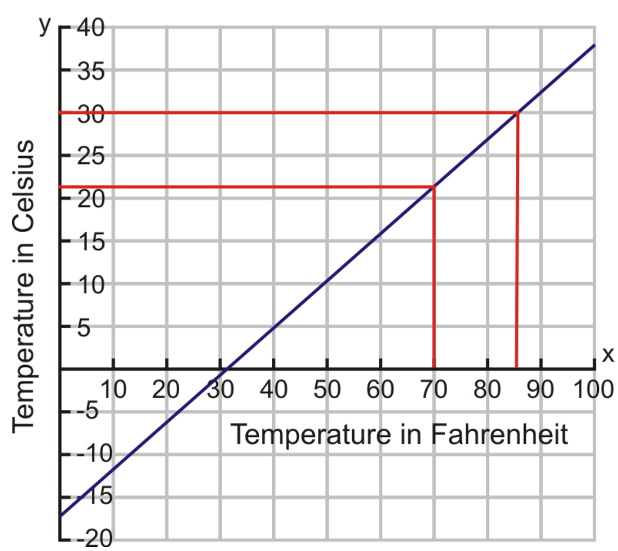
Below is a graph for converting marked prices in a downtown store into prices that include sales tax. Use the graph to determine the cost including sales tax for a \$6.00 pen in the store.



To find the relevant price with tax, first find the correct pre-tax price on the $x-$axis. This is the point $x = 6$.

Draw the line $x = 6$ up until it meets the function, then draw a horizontal line to the $y-$axis. This line hits at $y \approx 6.75$ (about three fourths of the way from $y = 6$ to $y = 7$).

The approximate cost including tax is \$6.75.



Example 4

The chart for converting temperature from Fahrenheit to Celsius is shown to the right. Use the graph to convert the following:

a) 70° Fahrenheit to Celsius

b) 0° Fahrenheit to Celsius

c) 30° Celsius to Fahrenheit

d) 0° Celsius to Fahrenheit

Solution

a) To find 70° Fahrenheit, we look along the Fahrenheit-axis (in other words the $x-$axis) and draw the line $x = 70$ up to the function. Then we draw a horizontal line to the Celsius-axis ($y-$axis). The horizontal line hits the axis at a little over 20 (21 or 22).

70° **Fahrenheit is approximately equivalent to** 21° **Celsius.**

b) To find 0° Fahrenheit, we just look at the $y-$axis. (Don't forget that this axis is simply the line $x = 0$.) The line hits the $y-$axis just below the half way point between -15 and -20.

0° **Fahrenheit is approximately equivalent to** $-18°$ **Celsius.**

c) To find 30° Celsius, we look up the Celsius-axis and draw the line $y = 30$ along to the function. When this horizontal line hits the function, we draw a line straight down to the Fahrenheit-axis. The line hits the axis at approximately 85.

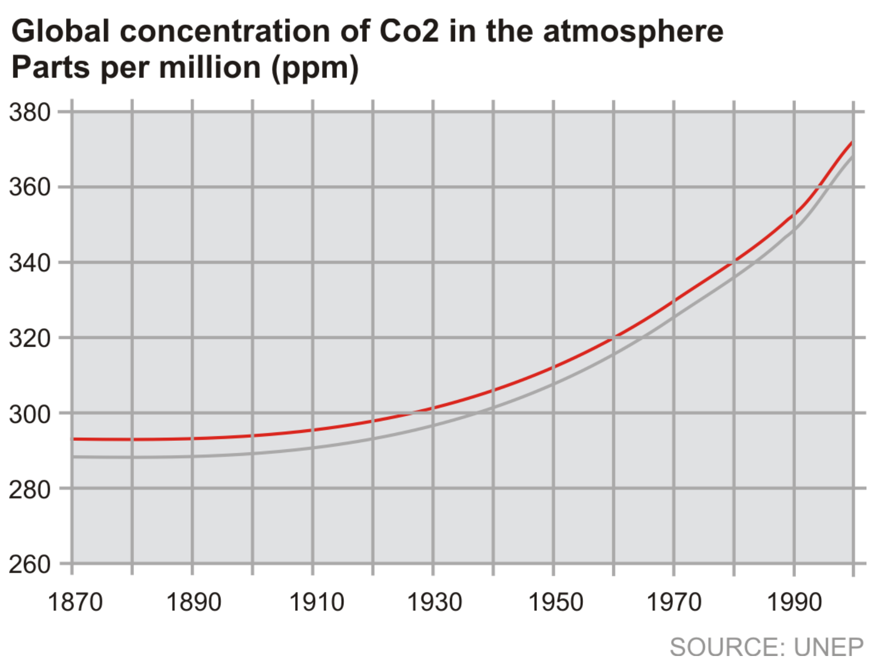
30° **Celsius is approximately equivalent to** 85° **Fahrenheit.**

d) To find 0° Celsius, we look at the Fahrenheit-axis (the line $y = 0$). The function hits the $x-$axis just right of 30.

0° **Celsius is equivalent to** 32° **Fahrenheit.**

Example 5

Here is an example of a graph you might see reported in the news. Most mainstream scientists believe that increased emissions of greenhouse cases, particularly carbon dioxide, are contributing to the warming of the planet. The graph below illustrates how carbon dioxide levels have increased as the world has industrialized.



From this graph, we can find the concentration of carbon dioxide found in the atmosphere in different years.

1900 - 285 parts per million

1930 - 300 parts per million

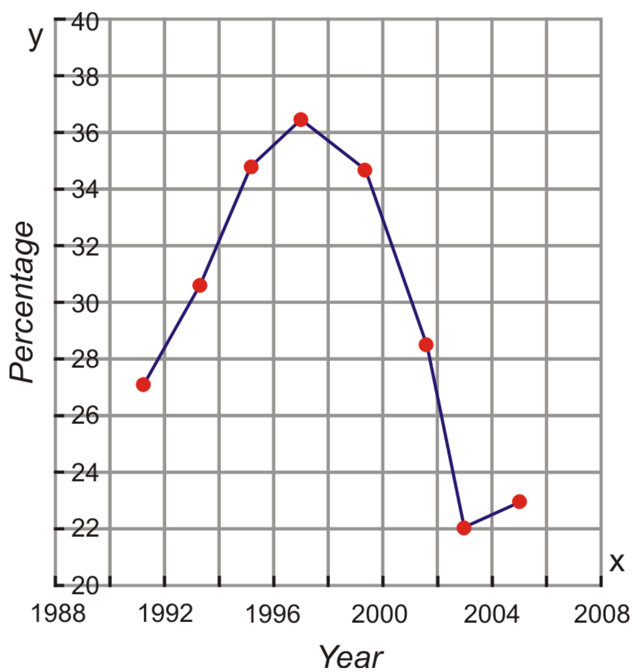
1950 - 310 parts per million

1990 - 350 parts per million

Practice Set

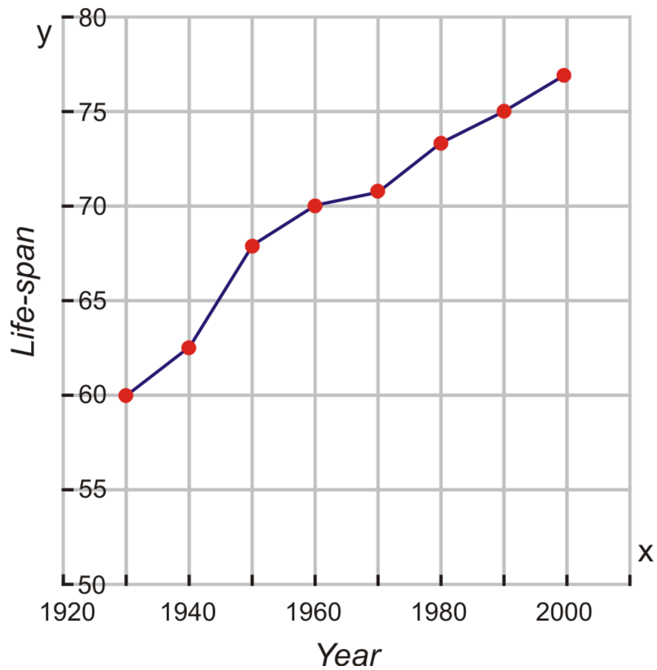
1. The students at a local high school took The Youth Risk Behavior Survey. The graph below shows the percentage of high school students who reported that they were current smokers. A person qualifies as a current smoker if he/she has smoked one or more cigarettes in the past 30 days. What percentage of high-school students were current smokers in the following years?

1. 1991
2. 1996
3. 2004
4. 2005



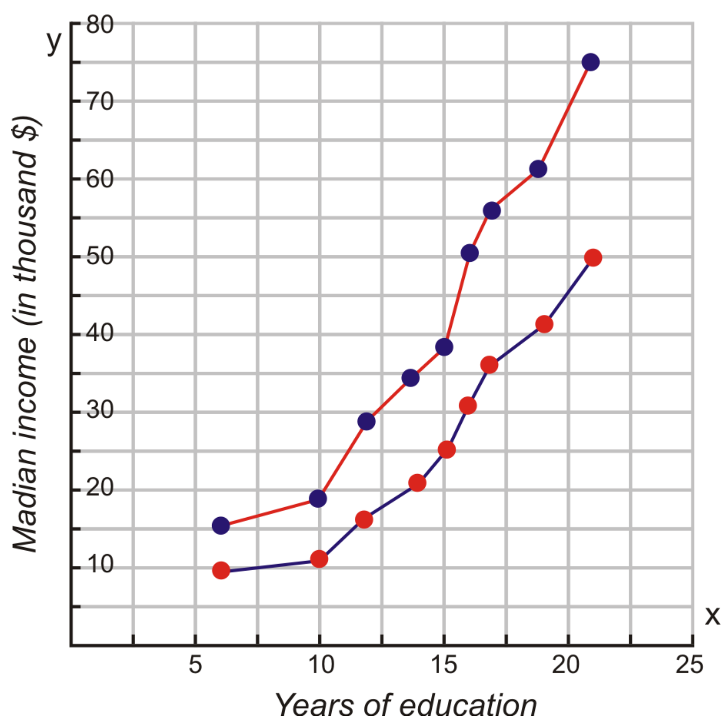
2. The graph below shows the average life-span of people based on the year in which they were born. This information comes from the National Vital Statistics Report from the Center for Disease Control. What is the average life-span of a person born in the following years?

1. 1940
2. 1955
3. 1980
4. 1995

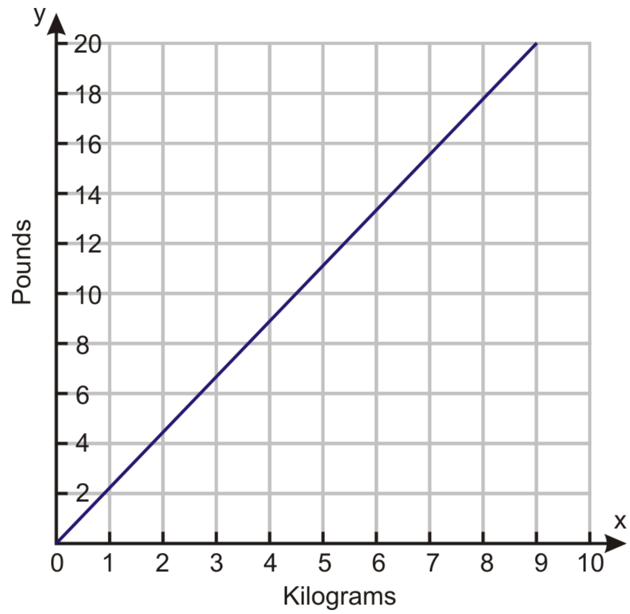


3. The graph below shows the median income of an individual based on his/her number of years of education. The top curve shows the median income for males and the bottom curve shows the median income for females. (Source: US Census, 2003.) What is the median income of a male who has the following years of education?

1. 10 years of education
2. 17 years of education
3. What is the median income of a female who has the same years of education?
4. 10 years of education
5. 17 years of education



4. The graph below shows a conversion chart for converting between weight in kilograms and weight in pounds. Use it to convert the following measurements.



1. (a) 4 kilograms into weight in pounds
   (b) 9 kilograms into weight in pounds
   (c) 12 pounds into weight in kilograms
   (d) 17 pounds into weight in kilograms

5. Use the graph from problem 4 to answer the following questions.

a. An employee at a sporting goods store is packing 3-pound weights into a box that can hold 8 kilograms. How many weights can she place in the box?b. After packing those weights, there is some extra space in the box that she wants to fill with one-pound weights. How many of those can she add?c. After packing those, she realizes she misread the label and the box can actually hold 9 kilograms. How many more one-pound weights can she add?

8.3 Graph a Linear Equation

In mathematics, we tend to use the words **formula** and **equation** to describe the rules we get when we express relationships algebraically. Interpreting and graphing these equations is an important skill that youll use frequently in math.

Example 1

A taxi costs more the further you travel. Taxis usually charge a fee on top of the per-mile charge to cover hire of the vehicle. In this case, the taxi charges $3 as a set fee and $0.80 per mile traveled. Here is the equation linking the cost in dollars (y) *to hire a taxi and the distance traveled in miles* (x).

$$y = 0.8x + 3$$

Graph the equation and use your graph to estimate the cost of a seven-mile taxi ride.

Solution

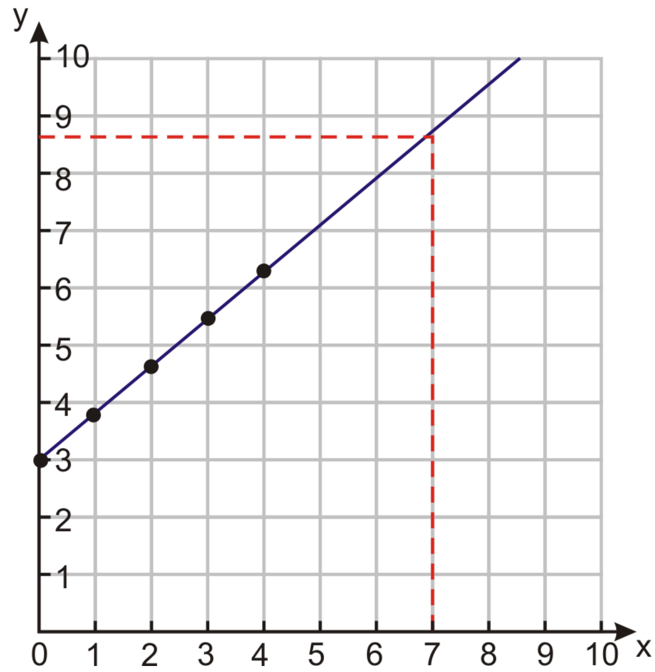
Well start by making a table of values. We will take a few values for x (0, 1, 2, 3, and 4), find the corresponding $y-$values, and then plot them. Since the question asks us to find the cost for a seven-mile journey, we need to choose a scale that can accommodate this.

First, heres our table of values:

TABLE 8.2:

x	y
0	3
1	3.8
2	4.6
3	5.4
4	6.2

And heres our graph:



To find the cost of a seven-mile journey, first we find $x = 7$ on the horizontal axis and draw a line up to our graph. Next, we draw a horizontal line across to the $y-$axis and read where it hits. It appears to hit around half way between $y = 8$ and $y = 9$. Let's call it 8.5.

A seven mile taxi ride would cost approximately \$8.50 (\$8.60 exactly).

Here are some things you should notice about this graph and the formula that generated it:

- The graph is a straight line (this means that the equation is **linear**).
- The graph crosses the $y-$axis at $y = 3$ (notice that theres $a + 3$ in the equationthats not a coincidence!). This is the base cost of the taxi.
- Every time we move **over** by one square we move **up** by 0.8 squares (notice that thats also the coefficient of x in the equation). This is the rate of charge of the taxi (cost per mile).
- If we move over by three squares, we move up by 3×0.8 squares.

Example 2

A small business has a debt of \$500,000 incurred from start-up costs. It predicts that it can pay off the debt at a rate of \$85,000 per year according to the following equation governing years in business (x) and debt measured in thousands of dollars (y).

$$y = -85x + 500$$

Graph the above equation and use your graph to predict when the debt will be fully paid.

Solution

First, we start with our table of values:

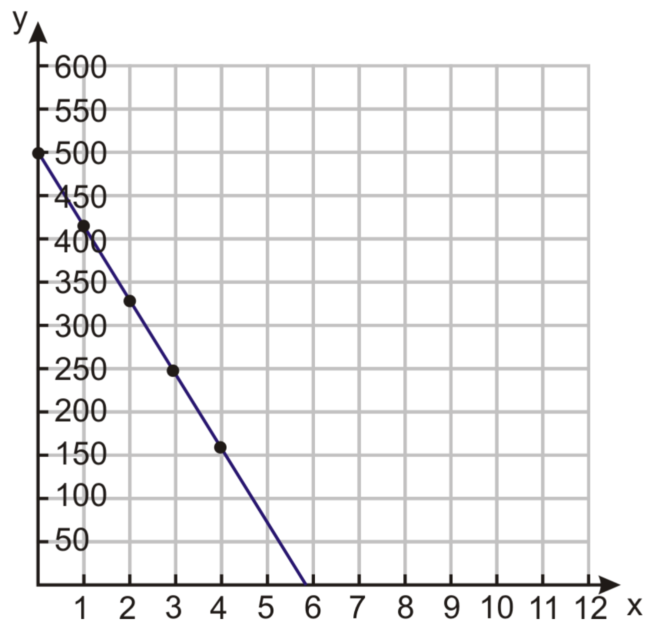
TABLE 8.3:

x	y
0	500
1	415

TABLE 8.3: (continued)

x	y
2	330
3	245
4	160

Then we plot our points and draw the line that goes through them:



Notice the scale weve chosen here. Theres no need to include any points above $y = 500$, but its still wise to allow a little extra.

Next we need to determine how many years it takes the debt to reach zero, or in other words, what $x-$value will make the $y-$value equal 0. We know its greater than four (since at $x = 4$ the $y-$value is still positive), so we need an $x-$scale that goes well past $x = 4$. Here weve chosen to show the $x-$values from 0 to 12, though there are many other places we could have chosen to stop.

To read the time that the debt is paid off, we simply read the point where the line hits $y = 0$ (the $x-$axis). It looks as if the line hits pretty close to $x = 6$. So **the debt will definitely be paid off in six years.**

To see more simple examples of graphing linear equations by hand, see the Khan Academy video on graphing lines at **http://tinyurl.com/7m2o2ya**. The narrator shows how to graph several linear equations, using a table of values to plot points and then connecting the points with a line.

Graphs and Equations of Horizontal and Vertical Lines

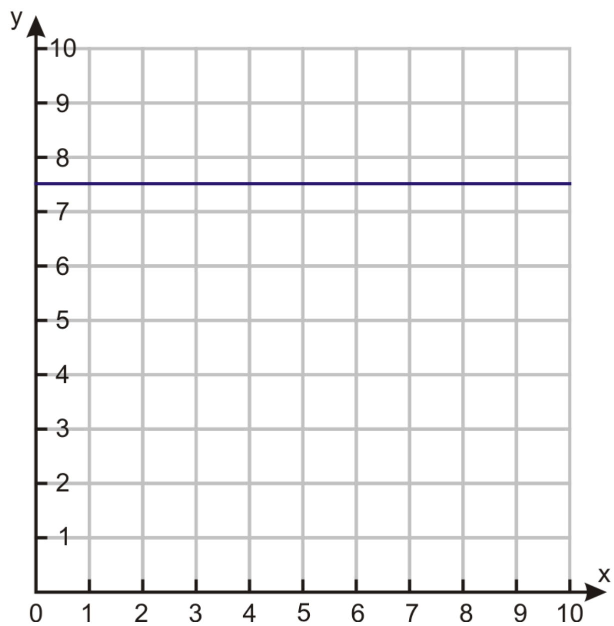
Example 3

Mad-cabs have an unusual offer going on. They are charging \$7.50 for a taxi ride of any length within the city limits. Graph the function that relates the cost of hiring the taxi (y) *to the length of the journey in miles* (x).

To proceed, the first thing we need is an **equation**. You can see from the problem that the cost of a journey doesnt depend on the length of the journey. It should come as no surprise that the equation then, does not have x in it. Since any value of x results in the same value of $y(7.5)$, the value you choose for x doesnt matter, so it isnt included in the equation. Here is the equation:

$y = 7.5$

The graph of this function is shown below. You can see that its simply a horizontal line.



Any time you see an equation of the form $y =$ constant, the graph is a horizontal line that intercepts the $y-$axis at the value of the constant.

Similarly, when you see an equation of the form $x =$ constant, then the graph is a vertical line that intercepts the $x-$axis at the value of the constant. (Notice that that kind of equation is a relation, and not a function, because each $x-$value (theres only one in this case) corresponds to many (actually an infinite number) $y-$values.)

Example 4

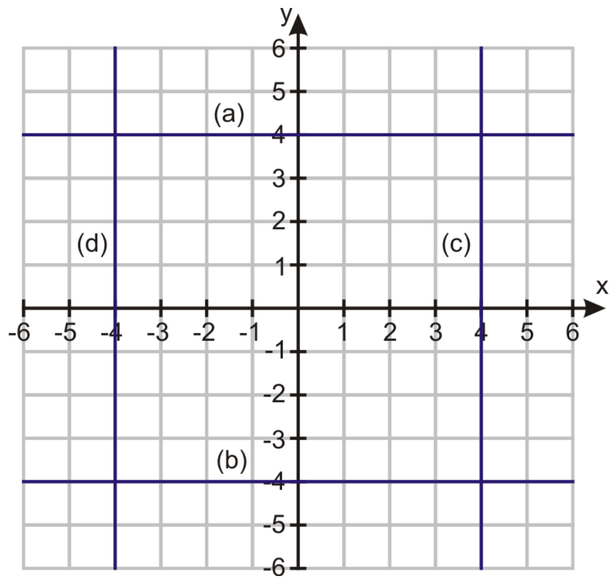
Plot the following graphs.

(a) $y = 4$

(b) $y = -4$

(c) $x = 4$

(d) $x = -4$



(a) $y = 4$ is a horizontal line that crosses the $y-$axis at 4.

(b) $y = -4$ is a horizontal line that crosses the $y-$axis at 4.

(c) $x = 4$ is a vertical line that crosses the $x-$axis at 4.

(d) $x = -4$ is a vertical line that crosses the $x-$axis at 4.

Example 5

Find an equation for the $x-$axis and the $y-$axis.

Look at the axes on any of the graphs from previous examples. We have already said that they intersect at the origin (the point where $x = 0$ and $y = 0$). The following definition could easily work for each axis.

*$x-$**axis:** A horizontal line crossing the $y-$axis at zero.*

*$y-$**axis:** A vertical line crossing the $x-$axis at zero.*

So using example 3 as our guide, we could define the $x-$axis as the line $y = 0$ and the $y-$axis as the line $x = 0$.

Lesson Summary

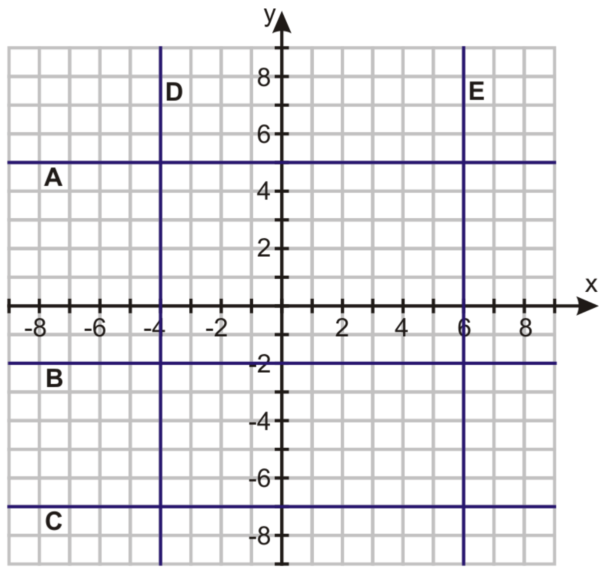
- Equations with the variables y and x can be graphed by making a chart of values that fit the equation and then plotting the values on a coordinate plane. This graph is simply another representation of the equation and can be analyzed to solve problems.
- Horizontal lines are defined by the equation $y =$ constant and vertical lines are defined by the equation $x =$ constant.
- Be aware that although we graph the function as a line to make it easier to interpret, the function may actually be discrete.

Practice Set

1. Make a table of values for the following equations and then graph them.
   (a) $y = 2x + 7$
   (b) $y = 0.7x - 4$
   (c) $y = 6 - 1.25x$
2. *Think of a number. Multiply it by 20, divide the answer by 9, and then subtract seven from the result.*

(a) Make a table of values and plot the function that represents this sentence.
(b) If you picked 0 as your starting number, what number would you end up with?
(c) To end up with 12, what number would you have to start out with?

3. Write the equations for the five lines (A through E) plotted in the graph below.



4. In the graph above, at what points do the following lines intersect?

(a) A and E
(b) A and D
(c) C and D
(d) B and the $y-$axis
(e) E and the $x-$axis
(f) C and the line $y = x$
(g) E and the line $y = \frac{1}{2}x$
(h) A and the line $y = x + 3$

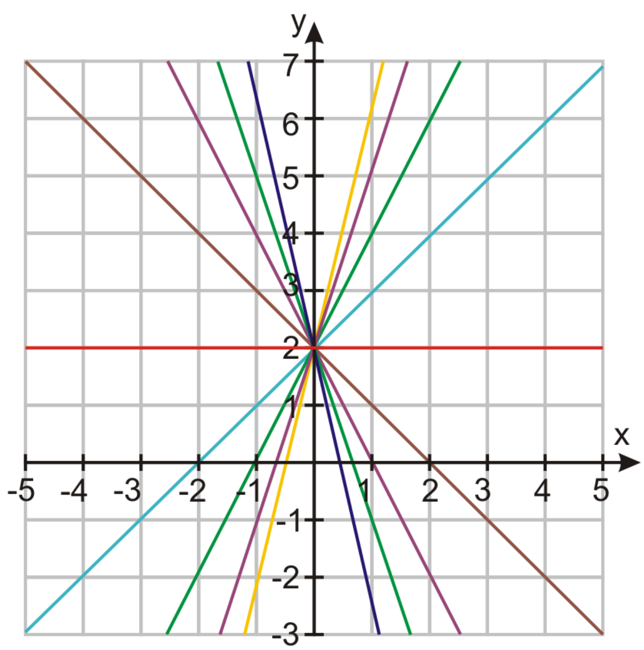
5. At the airport, you can change your money from dollars into euros. The service costs \$5, and for every additional dollar you get 0.7 euros.

(a) Make a table for this and plot the function on a graph.
(b) Use your graph to determine how many euros you would get if you give the office \$50.
(c) To get 35 euros, how many dollars would you have to pay?
(d) The exchange rate drops so that you can only get 0.5 euros per additional dollar. Now how many dollars do you have to pay for 35 euros?

8.4 Graphing Using Intercepts

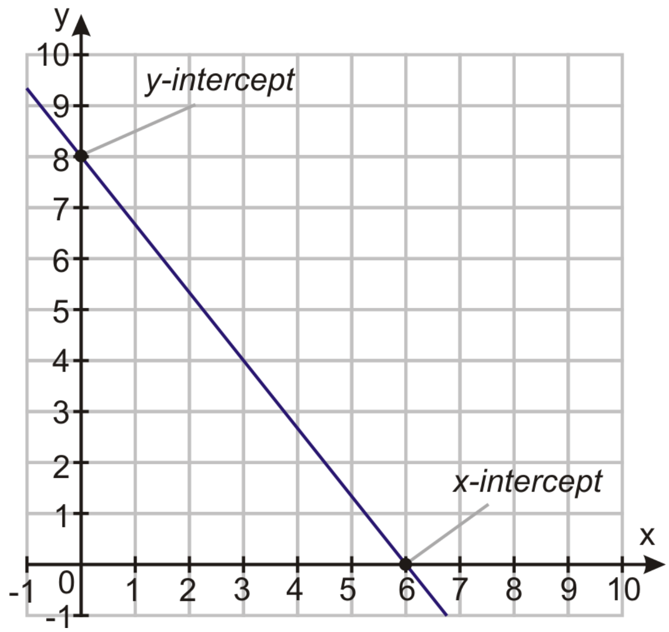
Introduction

Sanjits office is 25 miles from home, and in traffic he expects the trip home to take him an hour if he starts at 5 PM. Today he hopes to stop at the post office along the way. If the post office is 6 miles from his office, when will Sanjit get there?



If you know just one of the points on a line, youll find that isnt enough information to plot the line on a graph. As you can see in the graph above, there are many linesin fact, infinitely many linesthat pass through a single point. But what if you know two points that are both on the line? Then theres only one way to graph that line; all you need to do is plot the two points and use a ruler to draw the line that passes through both of them.

There are a lot of options for choosing which two points on the line you use to plot it. In this lesson, well focus on two points that are rather convenient for graphing: the points where our line crosses the $x-$and $y-$axes, or **intercepts.** Well see how to find intercepts algebraically and use them to quickly plot graphs.



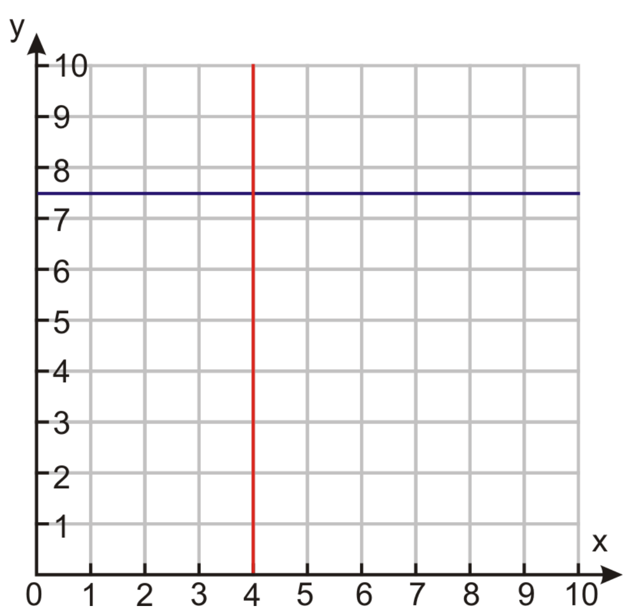
Look at the graph above. The $y-$**intercept** occurs at the point where the graph crosses the $y-$axis. The $y-$value at this point is 8, and the $x-$value is 0.

Similarly, the $x-$**intercept** occurs at the point where the graph crosses the $x-$axis. The $x-$value at this point is 6, and the $y-$value is 0.

So we know the coordinates of two points on the graph: (0, 8) and (6, 0). If wed just been given those two coordinates out of the blue, we could quickly plot those points and join them with a line to recreate the above graph.

Note: Not all lines will have both an $x-$and a $y-$intercept, but most do. However, horizontal lines never cross the $x-$axis and vertical lines never cross the $y-$axis.

For examples of these special cases, see the graph below.



Finding Intercepts

Example 1

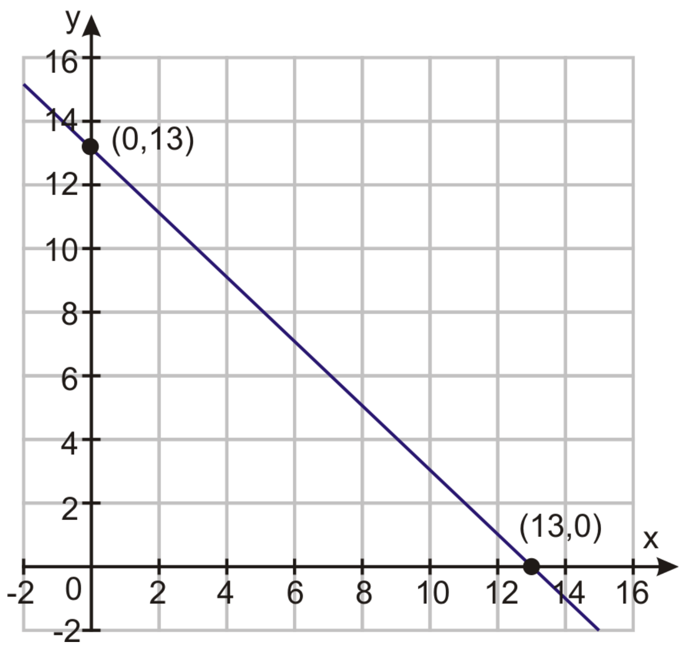
Find the intercepts of the line $y = 13 - x$ and use them to graph the function.

Solution

The first intercept is easy to find. The $y-$intercept occurs when $x = 0$. Substituting gives us $y = 13 - 0 = 13$, so the $y-$intercept is (0, 13).

Similarly, the $x-$intercept occurs when $y = 0$. Plugging in 0 for y gives us $0 = 13 - x$, solving this equation for x gives us $x = 13$. So (13, 0) is the $x-$intercept.

To draw the graph, simply plot these points and join them with a line.



Example 2

Graph the following equations by finding intercepts.

a) $y = 2x + 3$

b) $y = 7 - 2x$

c) $4x - 2y = 8$

d) $2x + 3y = -6$

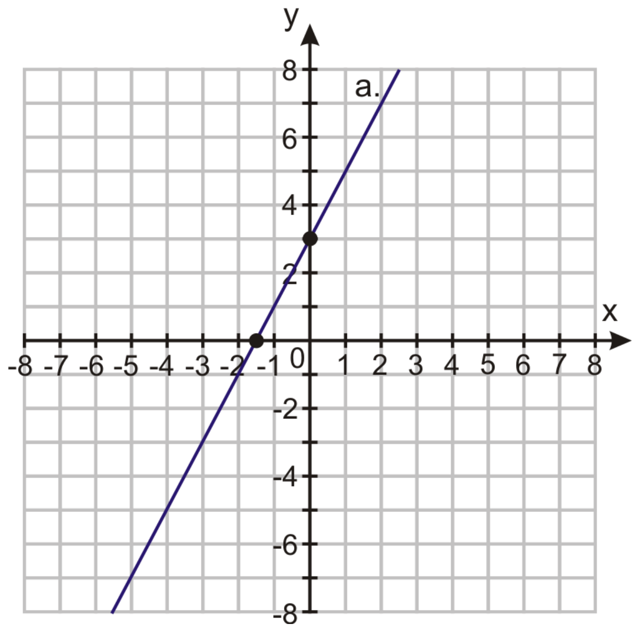
Solution

a) Find the $y-$intercept by plugging in $x = 0$:

$$y = 2 \cdot 0 + 3 = 3 \qquad -\text{the } y-\text{intercept is } (0,3)$$

Find the $x-$intercept by plugging in $y = 0$:

$$\begin{aligned} 0 &= 2x + 3 && -\textit{subtract } 3 \textit{ from both sides}: \\ -3 &= 2x && -\textit{divide by } 2: \\ -\frac{3}{2} &= x && -\text{the } x-\text{intercept is } (-1.5, 0) \end{aligned}$$

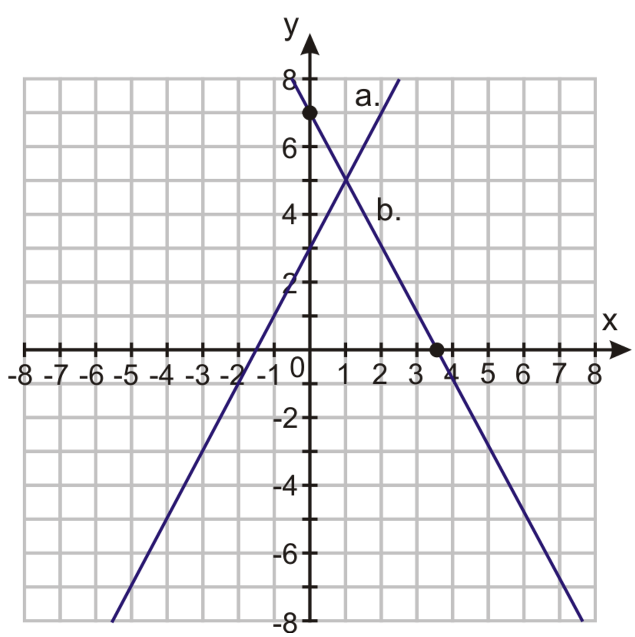


b) Find the $y-$intercept by plugging in $x = 0:$

$$y = 7 - 2 \cdot 0 = 7 \qquad - \text{the } y - \text{intercept is } (0,7)$$

Find the $x-$intercept by plugging in $y = 0:$

$$\begin{aligned} 0 &= 7 - 2x && - subtract\ 7\ from\ both\ sides: \\ -7 &= -2x && - divide\ by\ -2: \\ \frac{7}{2} &= x && - \text{the } x - \text{intercept is } (3.5,0) \end{aligned}$$

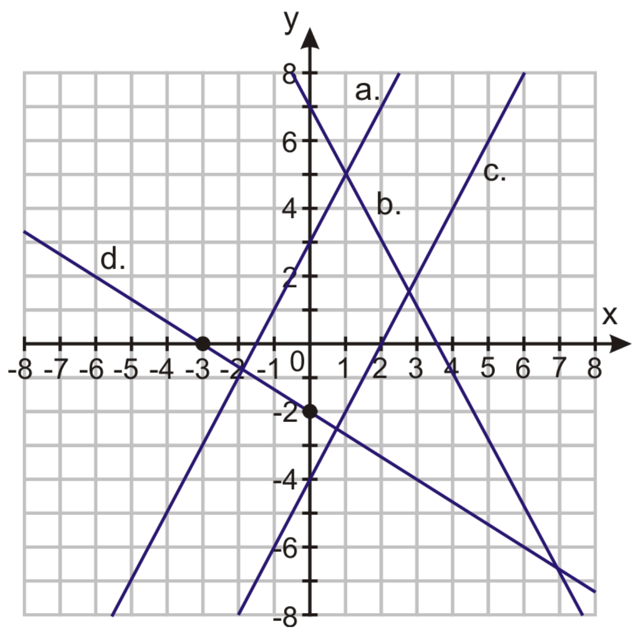


c) Find the $y-$intercept by plugging in $x = 0:$

$$\begin{aligned} 4 \cdot 0 - 2y &= 8 \\ -2y &= 8 && - \textit{divide by } -2 \\ y &= -4 && - \text{the } y - \text{intercept is } (0,-4) \end{aligned}$$

Find the $x-$intercept by plugging in $y = 0:$

$$\begin{aligned} 4x - 2 \cdot 0 &= 8 \\ 4x &= 8 && - \textit{divide by } 4: \\ x &= 2 && - \text{the } x - \text{intercept is } (2,0) \end{aligned}$$

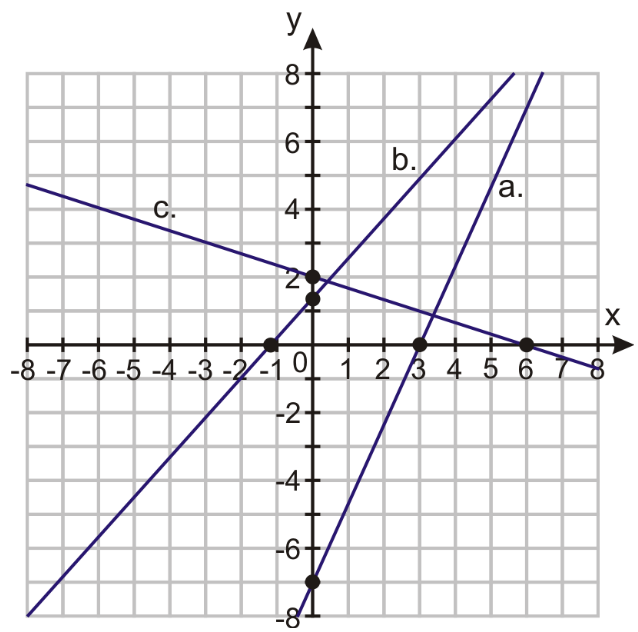


d) Find the $y-$intercept by plugging in $x = 0:$

$$\begin{aligned} 2 \cdot 0 + 3y &= -6 \\ 3y &= -6 && - \text{divide by } 3: \\ y &= -2 && - \text{the } y - \text{intercept is } (0,-2) \end{aligned}$$

Find the $x-$intercept by plugging in $y = 0:$

$$\begin{aligned} 2x + 3 \cdot 0 &= -6 \\ 2x &= -6 && - \text{divide by } 2: \\ x &= -3 && - \text{the } x - \text{intercept is } (-3,0) \end{aligned}$$



Finding Intercepts for General (Standard) Form Equations Using the Cover-Up Method

Look at the last two equations in the previous example. These equations are written in ***general form.*** General form equations are always written **coefficient** times x plus (or minus) **coefficient** times y equals **value**. In other words, they look like this:

$$ax+by=c$$

There is a neat method for finding intercepts in standard form, often referred to as the cover-up method.

Example 3

Find the intercepts of the following equations:

a) $7x-3y=21$

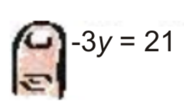
b) $12x-10y=-15$

c) $x+3y=6$

Solution

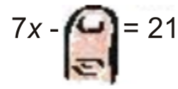
To solve for each intercept, we realize that at the intercepts the value of **either** x**or** y is zero, and so any terms that contain that variable effectively drop out of the equation. To make a term disappear, simply cover it (a finger is an excellent way to cover up terms) and solve the resulting equation.

a) To solve for the $y-$intercept we set $x=0$ and cover up the $x-$term:



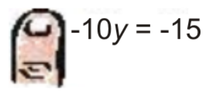
$-3y=21$
$y=-7 \qquad (0,-7)$ is the $y-$intercept.

Now we solve for the $x-$intercept:



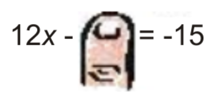
$7x = 21$
$x = 3 \qquad (3,0)$ is the $x-$intercept.

b) To solve for the $y-$intercept $(x = 0)$, cover up the $x-$term:



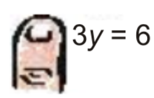
$-10y = -15$
$y = 1.5 \qquad (0,1.5)$ is the $y-$intercept.

Now solve for the $x-$intercept $(y = 0)$:



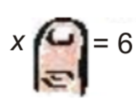
$12x = -15$
$x = -\frac{5}{4} \qquad (-1.25,0)$ is the $x-$intercept.

c) To solve for the $y-$intercept $(x = 0)$, cover up the $x-$term:



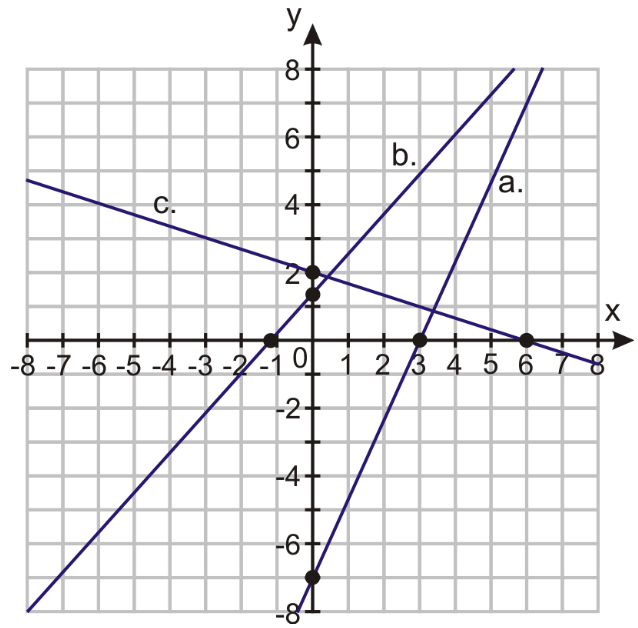
$3y = 6$
$y = 2 \qquad (0,2)$ is the $y-$intercept.

Solve for the $y-$intercept:



$x = 6 \qquad (6,0)$ is the $x-$intercept.

The graph of these functions and the intercepts is below:



To learn more about equations in standard form, try the Java applet at http://www.analyzemath.com/line/line.htm (scroll down and click the click here to start button.) You can use the sliders to change the values of a, b, and c and see how that affects the graph.

Solving Real-World Problems Using Intercepts of a Graph

Example 4

Jessie has $30 to spend on food for a class barbecue. Hot dogs cost $0.75 each (including the bun) and burgers cost $1.25 (including the bun). Plot a graph that shows all the combinations of hot dogs and burgers he could buy for the barbecue, without spending more than $30.

This time we will find an equation first, and then we can think logically about finding the intercepts.

If the number of burgers that Jessie buys is x, then the money he spends on burgers is $1.25x$

If the number of hot dogs he buys is y, then the money he spends on hot dogs is $0.75y$

So the total cost of the food is $1.25x + 0.75y$.

The total amount of money he has to spend is $30, so if he is to spend it ALL, we can use the following equation:

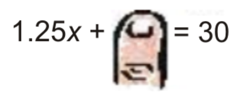
$$1.25x + 0.75y = 30$$

We can solve for the intercepts using the cover-up method. First the $y-$intercept

 $+ 0.75y = 30$

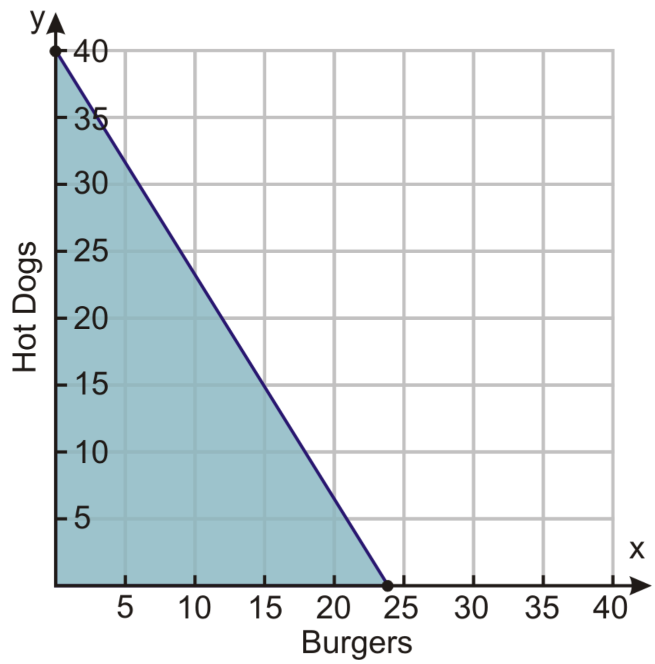
$0.75y = 30$
$y = 40 \qquad y-$intercept: $(0, 40)$

Then the $x-$intercept $(y = 0)$:



$1.25x = 30$
$x = 24 \qquad x-\text{intercept: } (24, 0)$

Now we plot those two points and join them to create our graph, shown here:



We could also have created this graph without needing to come up with an equation. We know that if John were to spend ALL the money on hot dogs, he could buy $\frac{30}{.75} = 40$ hot dogs. And if he were to buy only burgers he could buy $\frac{30}{1.25} = 24$ burgers. From those numbers, we can get 2 intercepts: (0 burgers, 40 hot dogs) and (24 burgers, 0 hot dogs). We could plot these just as we did above and obtain our graph that way.

As a final note, we should realize that Jesus problem is really an example of an inequality. He can, in fact, spend any amount up to \$30. The only thing he cannot do is spend more than \$30. The graph above reflects this: the line is the set of solutions that involve spending exactly \$30, and the shaded region shows solutions that involve spending less than \$30. Well work with inequalities some more in Chapter 6.

Lesson Summary

- A $y-$**intercept** occurs at the point where a graph crosses the $y-$axis (where $x = 0$) and an $x-$**intercept** occurs at the point where a graph crosses the $x-$axis (where $y = 0$).
- The $y-$intercept can be found by **substituting** $x = 0$ into the equation and solving for y. Likewise, the $x-$intercept can be found by **substituting** $y = 0$ into the equation and solving for x.
- Equations in general form can be solved for the intercepts by covering up the x (or y) term and solving the equation that remains.

Practice Set

1. Find the intercepts for the following equations using substitution.
   (a) $y = 3x - 6$

(b) $y = -2x + 4$
(c) $y = 14x - 21$
(d) $y = 7 - 3x$
(e) $y = 2.5x - 4$
(f) $y = 1.1x + 2.2$
(g) $y = \frac{3}{8}x + 7$
(h) $y = \frac{5}{9} - \frac{2}{7}x$

2. Find the intercepts of the following equations using the cover-up method.
   (a) $5x - 6y = 15$
   (b) $3x - 4y = -5$
   (c) $2x + 7y = -11$
   (d) $5x + 10y = 25$
   (e) $5x - 1.3y = 12$
   (f) $1.4x - 3.5y = 7$
   (g) $\frac{3}{5}x + 2y = \frac{2}{5}$
   (h) $\frac{3}{4}x - \frac{2}{3}y = \frac{1}{5}$
3. Use any method to find the intercepts and then graph the following equations.
   (a) $y = 2x + 3$
   (b) $6(x - 1) = 2(y + 3)$
   (c) $x - y = 5$
   (d) $x + y = 8$
4. At the local grocery store strawberries cost \$3.00 per pound and bananas cost \$1.00 per pound.
   (a) If I have \$10 to spend on strawberries and bananas, draw a graph to show what combinations of each I can buy and spend exactly \$10.
   (b) Plot the point representing 3 pounds of strawberries and 2 pounds of bananas. Will that cost more or less than \$10?
   (c) Do the same for the point representing 1 pound of strawberries and 5 pounds of bananas.
5. A movie theater charges \$7.50 for adult tickets and \$4.50 for children. If the theater takes in \$900 in ticket sales for a particular screening, draw a graph which depicts the possibilities for the number of adult tickets and the number of child tickets sold.
6. Why can't we use the intercept method to graph the following equation? $3(x + 2) = 2(y + 3)$
7. Name two more equations that we cant use the intercept method to graph.

CHAPTER 9 Introduction to Functions

Chapter Outline

Joseph decided to spend the day at the Theme Park. He could buy an all day pass, but he plans to ride only a few rides, so instead, he is going to pay by ride. Each ride costs \$2.00. To describe the amount of money Joseph will spend, several mathematical concepts can be used.

First, an expression can be written to describe the relationship between the cost per ride and the number of rides, r. An equation can also be written if the total amount he wants to spend is known. An inequality can be used if Joseph wanted to spend less than a certain amount.

9.1 Functions and Function Notation

Example 1: Using Joseph's situation, write the following:

a. An expression representing his total amount spent

b. An equation that shows Joseph wants to spend exactly \$22.00 on rides

c. An inequality that describes the fact that Joseph will not spend more than \$26.00 on rides

Solution: The variable in this situation is the number of rides Joseph will pay for. Call this r.

a. $2(r)$

b. $2(r) = 22$

c. $2(r) \leq 26$

In addition to an expression, equation, or inequality, Joseph's situation can be expressed in the form of a function.

Definition: A **function** is a relationship between two variables such that the input value has ONLY one output value.

Writing Equations as Functions

A function is a set of ordered pairs in which the first coordinate, the input, matches with exactly one second coordinate, the output. Equations that follow this definition can be written in function notation. The y coordinate represents the **dependent variable**, meaning the answers of this variable depend upon what is substituted for the other variable.

Consider Joseph's equation $m = 2r$. Using function notation, the value of the equation (the money spent m), is replaced with $f(r)$. f represents the function name and (r)represents the variable. In this case the parentheses do not mean multiplication – they separate the function name from the **independent variable.**

$$\begin{array}{c} input \\ \downarrow \\ \underbrace{f(x)}_{\substack{function \\ box}} = y \leftarrow output \end{array}$$

Example 2: Rewrite the following equations in function notation.

a. $y = 7x - 3$

b. $d = 65t$

c. $F = 1.8C + 32$

Solution:

a. According to the definition of a function, $y = f(x)$, so $f(x) = 7x - 3$.

b. This time the dependent variable is d. Function notation replaces the independent variable, so $d = f(t) = 65t$.

c. $F = f(C) = 1.8C + 32$

Why Use Function Notation?

Why is it necessary to use function notation? The necessity stems from using multiple equations. Function notation allows one to easily decipher between the equations. Suppose Joseph, Lacy, Kevin, and Alfred all went to the theme park together and chose to pay \$2.00 for each ride. Each person would have the same equation $m = 2r$. Without asking each friend, we could not tell which equation belonged to whom. By substituting function notation for the dependent variable, it is easy to tell which function belongs to whom. By using function notation, it will be much easier to graph multiple lines.

Example 3: Write functions to represent the total each friend spent at the park.

Solution: $J(r) = 2r$ represents Joseph's total, $L(r) = 2r$ represents Lacy, $K(r) = 2r$ represents Kevin, and $A(r) = 2r$ represents Alfred's total.

Using a Function to Generate a Table

A function really is a type of equation. Therefore, a table of values can be created by choosing values to represent the **independent variable.** The answers to each substitution represent $f(x)$.

Use Joseph's function to generate a table of values. Because the variable represents the number of rides Joseph will pay for, negative values do not make sense and are not included in the value of the independent variable.

TABLE 9.1:

r	$J(r) = 2r$
0	$2(0) = 0$
1	$2(1) = 2$
2	$2(2) = 4$
3	$2(3) = 6$
4	$2(4) = 8$
5	$2(5) = 10$
6	$2(6) = 12$

As you can see the list cannot include every possibility. A table allows for precise organization of data. It also provides an easy reference for looking up data and offers a set of coordinate points that can be plotted to create a graphical representation of the function. A table does have limitations; namely it cannot represent infinite amounts of data and does not always show the possibility of fractional values for the independent variable.

Domain and Range of a Function

The set of all possible input values for the independent variable is called the **domain.** The domain can be expressed in words, as a set, or as an inequality. The values resulting from the substitution of the domain represents the **range** of a function.

The domain of Joseph's situation will not include negative numbers because it does not make sense to ride negative rides. He also cannot ride a fraction of a ride, so decimals and fractional values do not make sense as input values. Therefore, the values of the independent variable r will be whole numbers beginning at zero.

Domain: All whole numbers

The values resulting from the substitution of whole numbers are whole numbers times two. Therefore, the **range** of Joseph's situation is still whole numbers just twice as large.

Range: All whole numbers

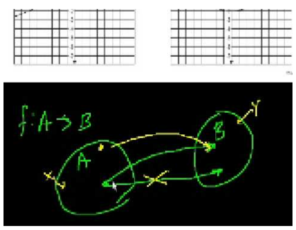
Example 4: A tennis ball is bounced from a height and bounces back to 75% of its previous height. Write its function and determine its domain and range.

Solution: The function of this situation is $h(b) = 0.75b$, where b represents the previous bounce height.

Domain: The previous bounce height can be any positive number, so $b \geq 0$.

Range: The new height is 75% of the previous height, and therefore will also be any positive number (decimal or whole number), so the range is **all positive real numbers.**

Multimedia Link For another look at the domain of a function, see the following video where the narrator solves a sample problem from the California Standards Test about finding the domain of an unusual function. KhanAcademy CA Algebra I Functions (6:34)



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Write a Function Rule

In many situations, data is collected by conducting a survey or an experiment. To visualize the data it is arranged into a table. Most often, a function rule is needed to predict additional values of the independent variable.

Example 5: *Write a function rule for the table.*

Number of CDs	2	4	6	8	10
Cost ($)	24	48	72	96	120

Solution: You pay \$24 for 2 CDs, \$48 for 4 CDs, \$120 for 10 CDs. That means that each CD costs \$12.

We can write the function rule.

Cost $= \$12 \times$ number of CDs or $f(x) = 12x$

Example 6: *Write a function rule for the table.*

$amp;x$	-3	-2	-1	0	1	2	3
$amp;y$	3	2	1	0	1	2	3

Solution: The values of the dependent variable are always the positive outcomes of the input values. This relationship has a special name, the absolute value. The function rule looks like this: $f(x) = |x|$.

Represent a Real-World Situation with a Function

Let's look at a real-world situation that can be represented by a function.

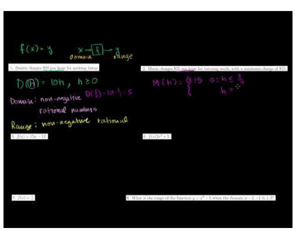
Example 7: *Maya has an internet service that currently has a monthly access fee of $11.95 and a connection fee of $0.50 per hour. Represent her monthly cost as a function of connection time.*

Solution: Let $x =$ the number of hours Maya spends on the internet in one month and let $y =$ Maya's monthly cost. The monthly fee is $11.95 with an hourly charge of $0.50.

The total cost = flat fee + hourly fee × number of hours. The function is $y = f(x) = 11.95 + 0.50x$

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. CK-12 Basic Algebra:Domain and Range ofa Function (12:52)



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1. Rewrite using function notation: $y = \frac{5}{6}x - 2$.
2. What is one benefit of using function notation?
3. Define *domain.*
4. *True or false.* Range is the set of all possible inputs for the independent variable.
5. Generate a table from $-5 \le x \le 5$ for $f(x) = -(x)^2 - 2$
6. Use the following situation for question 6: *Sheri is saving for her first car. She currently has $515.85 and is savings $62 each week.*
   (a) Write a function rule for the following situation:
   (b) Can the domain be "all real numbers?" Explain your thinking.
   (c) How many weeks would it take Sheri to save $1,795.00?

Identify the domain and range of the following functions.

7. Dustin charges \$10 per hour for mowing lawns.
8. Maria charges \$25 per hour for tutoring math, with a minimum charge of \$15.
9. $f(x) = 15x - 12$
10. $f(x) = 2x^2 + 5$
11. $f(x) = \frac{1}{x}$
12. What is the range of the function $y = x^2 - 5$ when the domain is -2, -1, 0, 1, 2?
13. What is the range of the function $y = 2x - \frac{3}{4}$ when the domain is -2.5, 1.5, 5?
14. Angie makes \$6.50 per hour working as a cashier at the grocery store. Make a table of values that shows her earning for input values 5, 10, 15, 20, 25, 30.
15. The area of a triangle is given by: $A = \frac{1}{2}bh$. If the height of the triangle is 8 centimeters, make a table of values that shows the area of the triangle for heights 1, 2, 3, 4, 5, and 6 centimeters.
16. Make a table of values for the function $f(x) = \sqrt{2x+3}$ for input values -1, 0, 1, 2, 3, 4, 5.
17. Write a function rule for the table

$amp;input$	3	4	5	6
$amp;output$	9	16	25	36

18. Write a function rule for the table

hours	0	1	2	3
cost	15	20	25	30

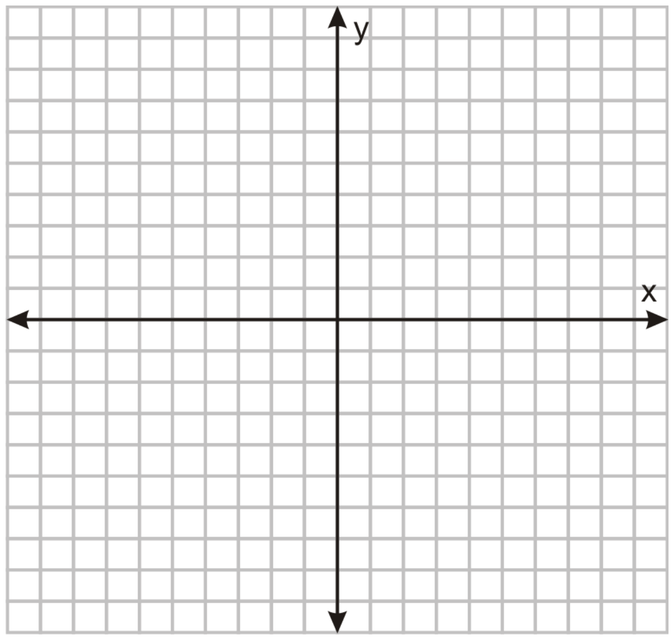
19. Write a function rule for the table

$amp;input$	0	1	2	3
$amp;output$	24	12	6	3

20. Write a function that represents the number of cuts you need to cut a ribbon in x number of pieces.
21. Solomon charges a \$40 flat rate and \$25 per hour to repair a leaky pipe. Write a function that represents the total fee charge as a function of hours worked. How much does Solomon earn for a 3 hour job?
22. Rochelle has invested \$2500 in a jewelry making kit. She makes bracelets that she can sell for \$12.50 each. How many bracelets does Rochelle need to make before she breaks even?
23. Make up a situation in which the domain is all real numbers but the range is all whole numbers.

9.2 Functions as Graphs

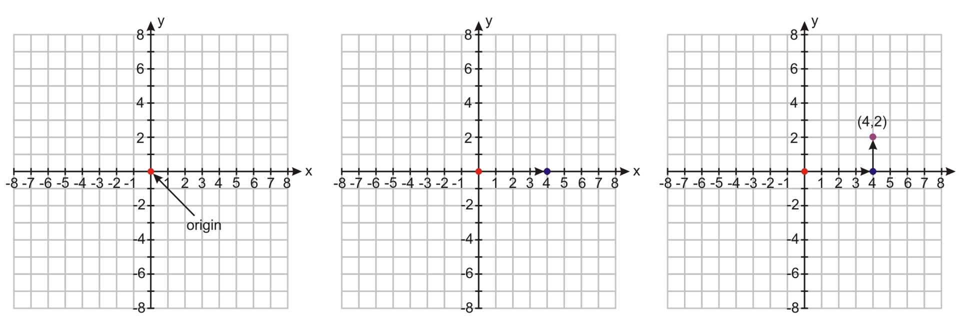
Once a table has been created for a function, the next step is to visualize the relationship by graphing the coordinates *(independent value, dependent value).* In previous courses, you have learned how to plot ordered pairs on a coordinate plane. The first coordinate represents the horizontal distance from the origin (the point where the axes intersect). The second coordinate represents the vertical distance from the origin.



To graph a coordinate point such as (4,2) we start at the origin.

Because the first coordinate is positive four, we move 4 units to the right.

From this location, since the second coordinate is positive two, we move 2 units up.



Example 1: *Plot the following coordinate points on the Cartesian plane.*

(a) (5, 3)

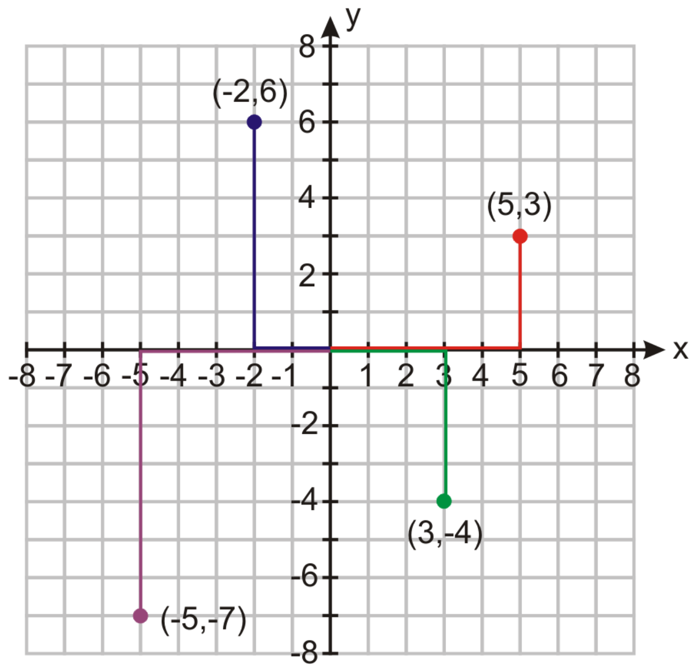
(b) (-2, 6)

(c) (3, -4)

(d) (-5, -7)

Solution: We show all the coordinate points on the same plot.

Notice that:



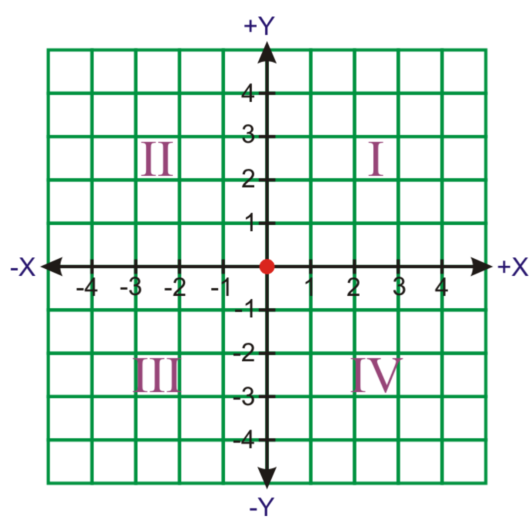
For a positive x value we move to the right.

For a negative x value we move to the left.

For a positive y value we move up.

For a negative y value we move down.

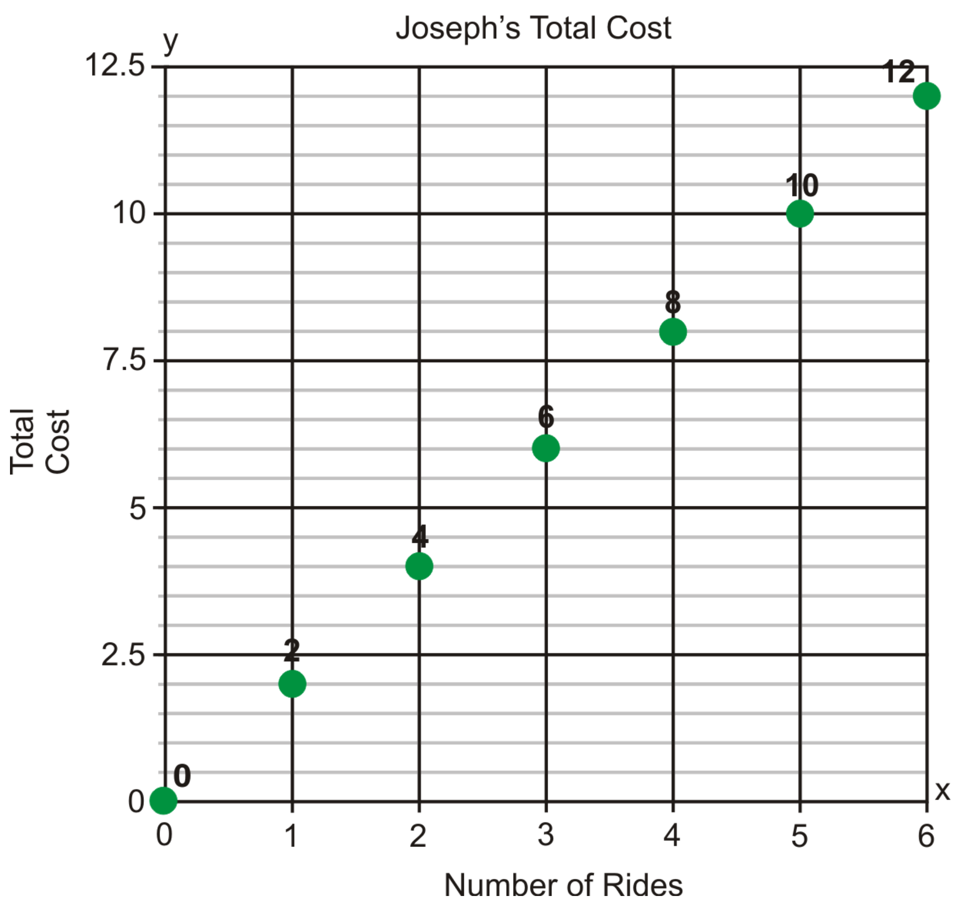
When referring to a coordinate plane, also called a Cartesian plane, the four sections are called **quadrants.** The first quadrant is the upper right section, the second quadrant is the upper left, the third quadrant is the lower left and the fourth quadrant is the lower right.



Suppose we wanted to visualize Joseph's total cost of riding at the park. Using the table below, the graph can be constructed as (number of rides, total cost)

TABLE 9.2:

r	$J(r) = 2r$
0	$2(0) = 0$
1	$2(1) = 2$
2	$2(2) = 4$
3	$2(3) = 6$
4	$2(4) = 8$
5	$2(5) = 10$
6	$2(6) = 12$



The green dots represent the combination of $(r, J(r))$. The dots are not connected because the domain of this function is all whole numbers. By connecting the points we are indicating that all values between the ordered pairs are also solutions to this function. Can Joseph ride $2\frac{1}{2}$ rides? Of course not! Therefore, we leave this situation as a **scatter plot.**

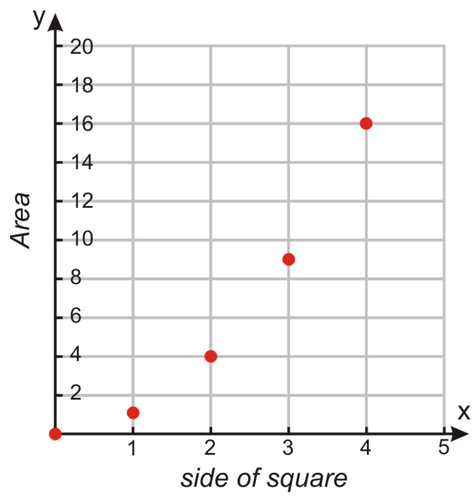
Example 2: Graph the function that has the following table of values.

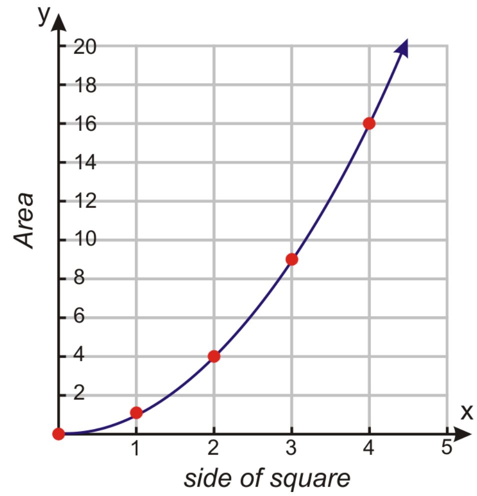
Side of the Square	0	1	2	3	4
Area of the Square	0	1	4	9	16

Solution: The table gives us five sets of coordinate points:

(0, 0), (1, 1), (2, 4), (3, 9), (4, 16).

To graph the function, we plot all the coordinate points. Because the length of a square can be fractional values, but not negative, the domain of this function is all positive real numbers, or $x \geq 0$. This means the ordered pairs can be connected with a smooth curve. It will continue forever in the positive direction, shown by an arrow.



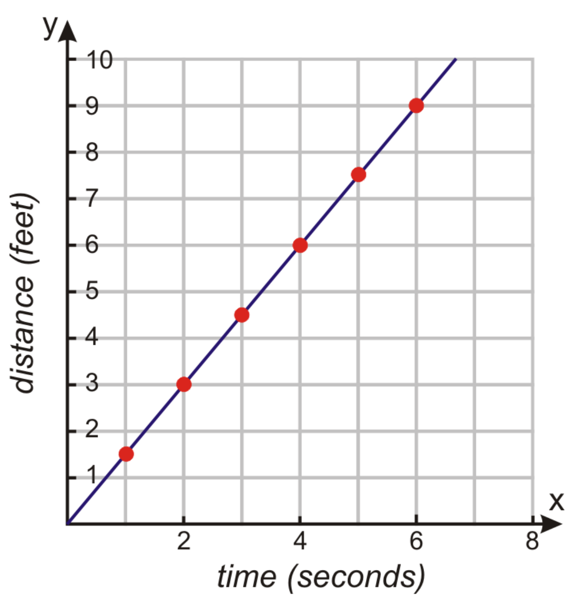


Writing a Function Rule Using a Graph

In many cases you are given a graph and asked to determine its function. From a graph, you can read pairs of coordinate points that are on the curve of the function. The coordinate points give values of dependent and independent variables. These variables are related to each other by a rule. It is important we make sure this rule works for all the points on the curve.

In this course you will learn to recognize different kinds of functions. There will be specific methods that you can use for each type of function that will help you find the function rule. For now, we will look at some basic examples and find patterns that will help us figure out the relationship between the dependent and independent variables.

Example 3: The graph below shows the distance that an inchworm covers over time. Find the function rule that shows how distance and time are related to each other.



Solution: Make table of values of several coordinate points to identify a pattern.

Time	0	1	2	3	4	5	6
Distance	0	1.5	3	4.5	6	7.5	9

We can see that for every second the distance increases by 1.5 feet. We can write the function rule as:

Distance $= 1.5 \times$ time

The equation of the function is $f(x) = 1.5x$

Determining Whether a Relation is a Function

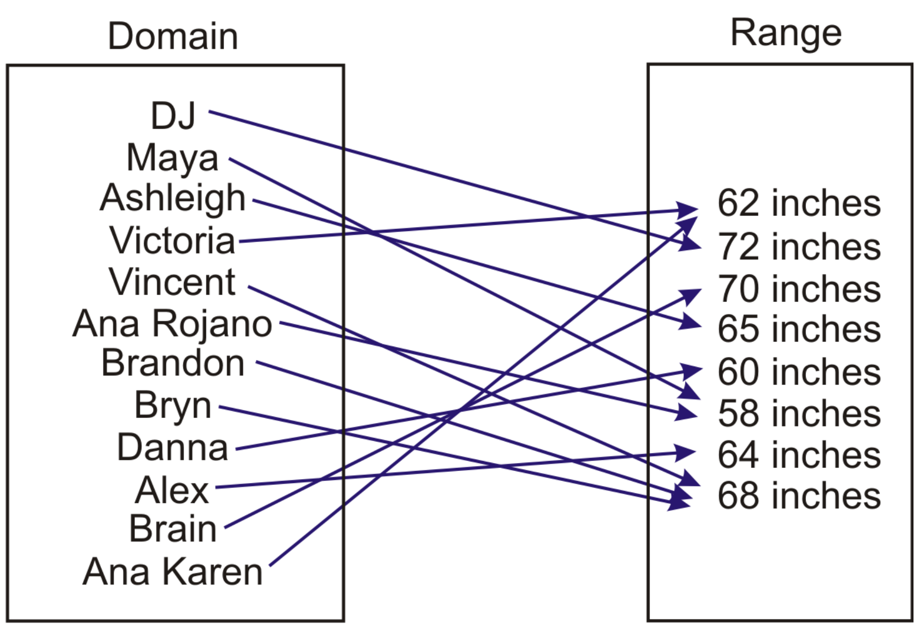
You saw that a function is a **relation** between the independent and the dependent variables. It is a rule that uses the values of the independent variable to give the values of the dependent variable. A function rule can be expressed in words, as an equation, as a table of values and as a graph. All representations are useful and necessary in understanding the relation between the variables.

Definition: A **relation** is a set of ordered pairs. Mathematically, a function is a special kind of relation.

Definition: A **function** is a relation between two variables such that each input value has EXACTLY one output value.

This usually means that each $x-$value has only one $y-$value assigned to it. But, not all functions involve x and y.

Consider the relation that shows the heights of all students in a class. The domain is the set of people in the class and the range is the set of heights. Each person in the class cannot be more than one height at the same time. This relation is a function because for each person there is exactly one height that belongs to him or her.



Notice that in a function, a value in the range can belong to more than one element in the domain, so more than one person in the class can have the same height. The opposite is not possible, one person cannot have multiple heights.

Example 4: *Determine if the relation is a function.*

a) (1, 3), (-1, -2), (3, 5), (2, 5), (3, 4)

b) (-3, 20), (-5, 25), (-1, 5), (7, 12), (9, 2)

Solution:

a) To determine whether this relation is a function, we must follow the definition of a function. Each $x-$coordinate can have ONLY one $y-$coordinate. However, since the $x-$coordinate of 3 has two $y-$coordinates, 4 and 5, this relation is NOT a function.

b) Applying the definition of a function, each $x-$coordinate has only one $y-$coordinate. Therefore, this relation is a function.

Determining Whether a Graph is a Function

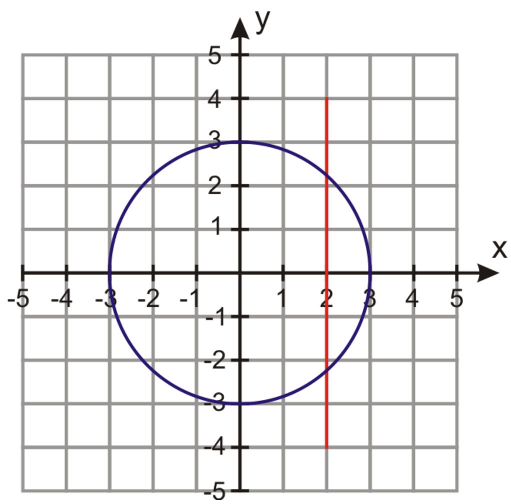
One way to determine whether a relation is a function is to construct a **flow chart** linking each dependent value to its matching independent value. Suppose, however, all you are given is the graph of the relation. How can you determine whether it is a function?

You could organize the ordered pairs into a table or a flow chart, similar to the student and height situation. This could be a lengthy process, but it is one possible way. A second way is to use the **Vertical Line Test.** Applying this test gives a quick and effective visual to decide if the graph is a function.

Theorem: Part A) A relation is a function if there are no vertical lines that intersect the graphed relation in more than one point.

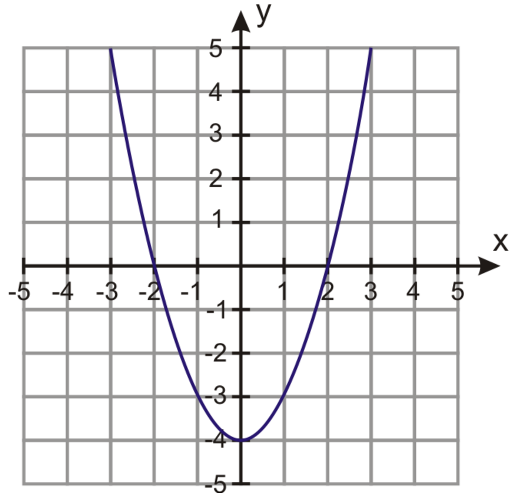
Part B) If a graphed relation does not intersect a vertical line in more than one point, then that relation is a function.

Is this graphed relation a function?



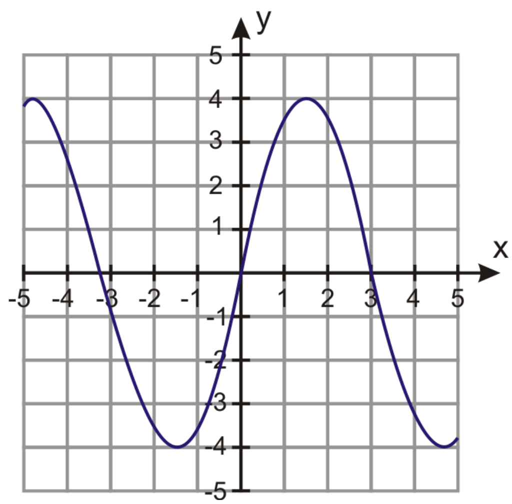
By drawing a vertical line (the red line) through the graph, we can see that the vertical line intersects the circle more than once. Therefore, this graph is NOT a function.

Here is a second example:



No matter where a vertical line is drawn through the graph, there will be only one intersection. Therefore, this graph is a function.

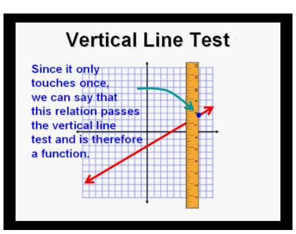
Example 4: *Determine if the relation is a function.*



Solution: Using the Vertical Line Test, we can conclude the relation is a function.

For more information:

Watch this YouTube video giving step-by-step instructions of the Vertical Line Test. CK-12 Basic Algebra:Vertical Line Test (3:11)



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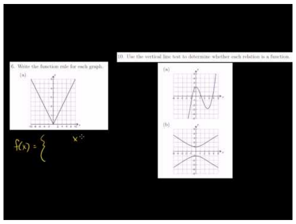
Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. CK-12 Basic Algebra:Functions as Graphs (9:34)



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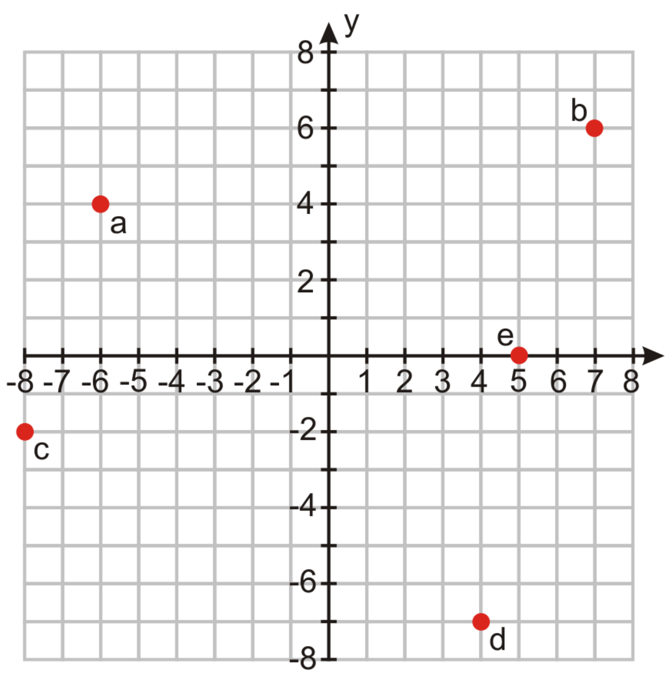
In 1 – 5, plot the coordinate points on the Cartesian plane.

1. (4, -4)
2. (2, 7)
3. (-3, -5)
4. (6, 3)
5. (-4, 3)

Using the coordinate plane below, give the coordinates for a - e.

6. 

In 7 – 9, graph the relation on a coordinate plane. According to the situation, determine whether to connect the ordered pairs with a smooth curve or leave as a scatterplot.

7.

X	-10	-5	0	5	10
Y	-3	-0.5	2	4.5	7

TABLE 9.3:

Side of cube (in inches)	Volume of cube (in inches^3)
0	0
1	1
2	8
3	27
4	64

TABLE 9.4:

Time (in hours)	Distance (in miles)
-2	-50
-1	25
0	0
1	5
2	50

In 10 – 12, graph the function.

10. Brandon is a member of a movie club. He pays a $50 annual membership and $8 per movie.
11. $f(x) = (x-2)^2$
12. $f(x) = 3.2^x$

In 13 – 16, determine if the relation is a function.

13. (1, 7), (2, 7), (3, 8), (4, 8), (5, 9)
14. (1, 1), (1, -1), (4, 2), (4, -2), (9, 3), (9, -3)

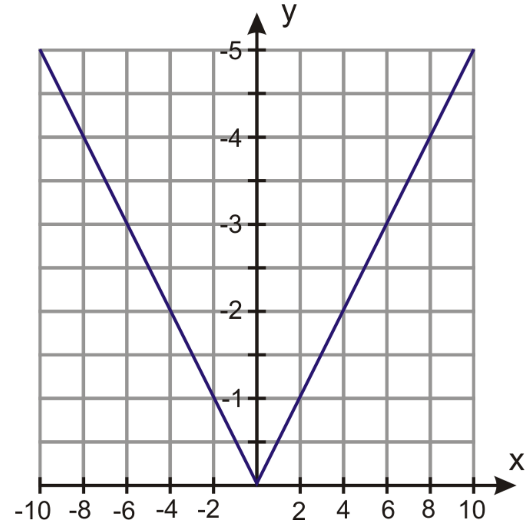
15.

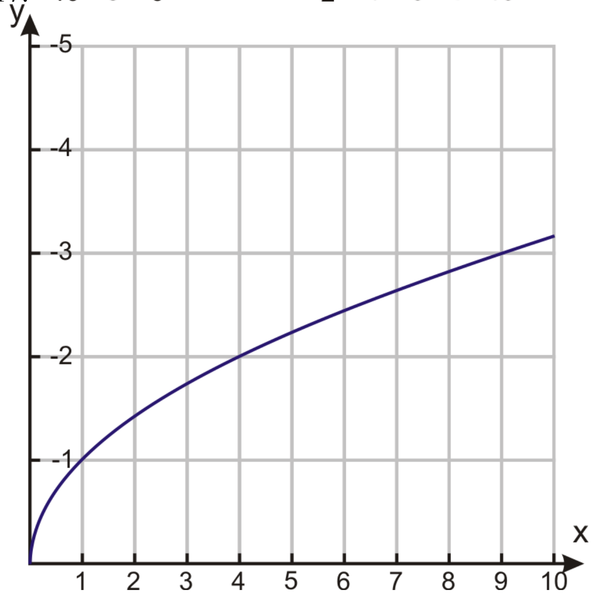
Age	20	25	25	30	35
Number of jobs by that age	3	4	7	4	2

16.

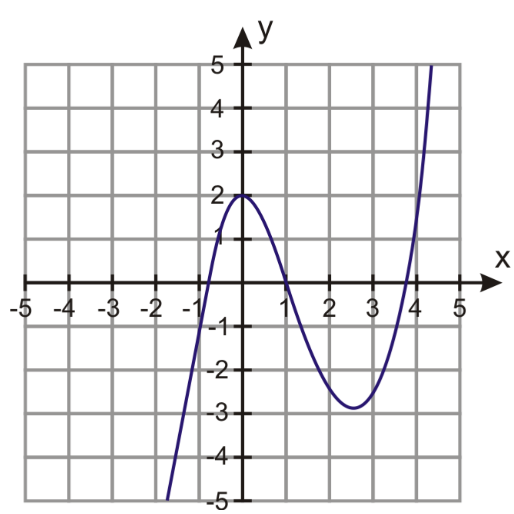
$amp;x$	-4	-3	-2	-1	0
$amp;y$	16	9	4	1	0

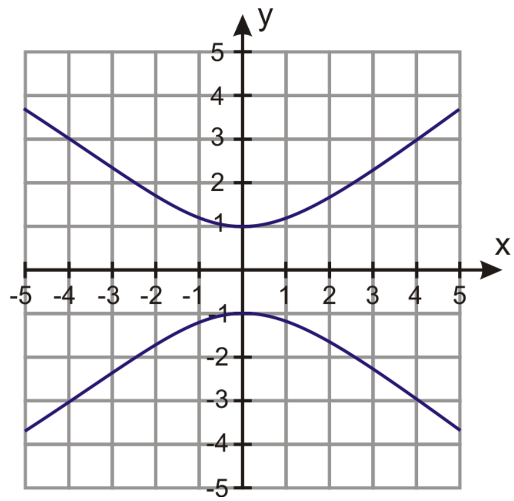
In 17 and 18, write a function rule for the graphed relation.

17. 

18. 

In 22 – 23, determine whether the graphed relation is a function.

22. 

23. 

9.3 Using Function Notation

Function notation allows you to easily see the input value for the independent variable inside the parentheses.

Example: Consider the function $f(x) = -\frac{1}{2}x^2$

Evaluate $f(4)$.

Solution: The value inside the parentheses is the value of the variable x. Use the Substitution Property to evaluate the function for $x = 4$.

$$\begin{aligned} f(4) &= -\frac{1}{2}(4^2) \\ f(4) &= -\frac{1}{2} \cdot 16 \\ f(4) &= -8 \end{aligned}$$

To use function notation, the equation must be written in terms of x. This means that the $y-$variable must be isolated on one side of the equal sign.

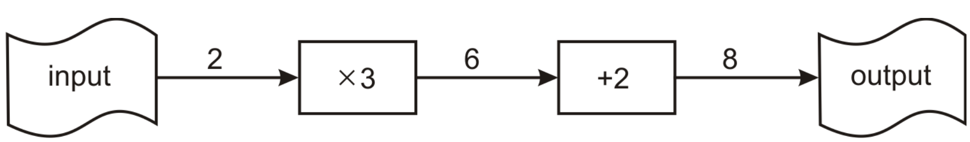
Example: Rewrite $9x + 3y = 6$ using function notation.

Solution: The goal is to rearrange this equation so the equation looks like $y =$. Then replace $y = with\ f(x) =$.

$$\begin{aligned} 9x + 3y &= 6 && \text{Subtract } 9x \text{ from both sides.} \\ 3y &= 6 - 9x && \text{Divide by 3.} \\ y &= \frac{6 - 9x}{3} = 2 - 3x \\ f(x) &= 2 - 3x \end{aligned}$$

Functions as Machines

You can think of a function as a machine. You start with an input (some value), the machine performs the operations (it does the work), and your output is the answer. For example, $f(x) = 3x + 2$ takes ***some number***, x, multiplies it by 3 and adds two. As a machine, it would look like this:



When you use the function machine to evaluate $f(2)$, the solution is $f(2) = 8$.

Example 1: A function is defined as $f(x) = 6x - 36$. Determine the following:

a) $f(2)$

b) $f(p)$

Solution:

a) Substitute $x = 2$ into the function $f(x)$: $f(2) = 6 \cdot 2 - 36 = 12 - 36 = -24$

b) Substitute $x = p$ into the function $f(x)$: $f(p) = 6p + 36$

Practice Set

For each of the following functions evaluate $f(-3); f(7); f(0), \textit{and } f(z)$:

1. $f(x) = -2x + 3$
2. $f(x) = 0.7x + 3.2$
3. $f(x) = \frac{5(2-x)}{11}$
4. $f(t) = \frac{1}{2}t^2 + 4$
5. $f(x) = 3 - \frac{1}{2}x$
6. The roasting guide for a turkey suggests cooking for 100 minutes plus an additional 8 minutes per pound.
   (a) Write a function for the roasting time, given the turkey weight in pounds (x).
   (b) Determine the time needed to roast a 10-lb turkey.
   (c) Determine the time needed to roast a 27-lb turkey.
   (d) Determine the maximum size turkey you could roast in $4\frac{1}{2}$ hours.
7. $F(C) = 1.8C + 32$ is the function used to convert Celsius to Fahrenheit. Find $F(100)$ and explain what it represents.
8. A prepaid phone card comes with \$20 worth of calls. Calls cost a flat rate of \$0.16 per minute. Write the value of the card as a function of minutes per calls. Use a function to determine the number of minutes of phone calls you can make with the card.
9. You can burn 330 calories during one hour of bicycling. Write this situation using $b(h)$ as the function notation. Evaluate $b(0.75)$ and explain its meaning.
10. Sadie has a bank account with a balance of \$650.00. She plans to spend \$55 per week.
    (a) Write this using function notation.
    (b) Evaluate her account after 10 weeks. What can you conclude?

CHAPTER 10 Linear Functions

Chapter Outline

10.1 Slope and Rate of Change

Introduction

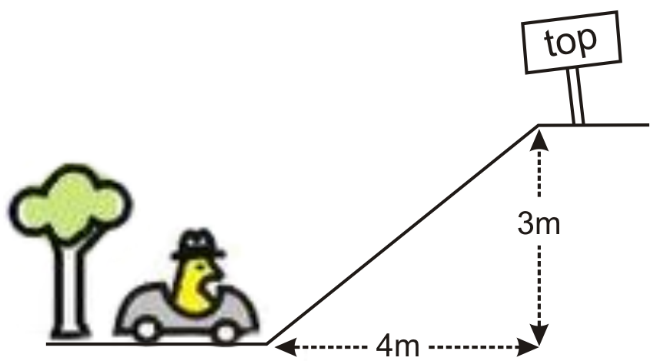
Wheelchair ramps at building entrances must have a slope between $\frac{1}{16}$ and $\frac{1}{20}$. If the entrance to a new office building is 28 inches off the ground, how long does the wheelchair ramp need to be?

We come across many examples of slope in everyday life. For example, a slope is in the pitch of a roof, the grade or incline of a road, or the slant of a ladder leaning on a wall. In math, we use the word **slope** to define steepness in a particular way.

$$\text{Slope} = \frac{\text{distance moved vertically}}{\text{distance moved horizontally}}$$

To make it easier to remember, we often word it like this:

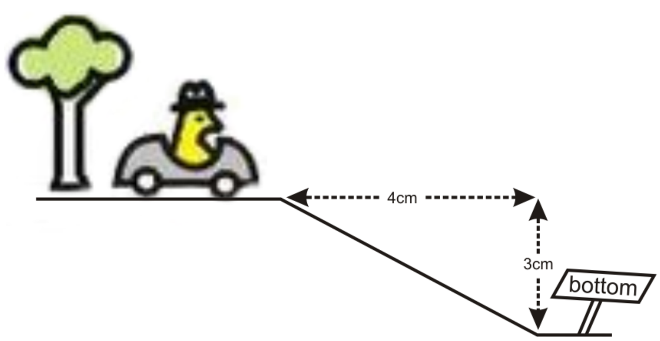
$$\text{Slope} = \frac{\text{rise}}{\text{run}}$$



In the picture above, the slope would be the ratio of the **height** of the hill to the horizontal **length** of the hill. In other words, it would be $\frac{3}{4}$, or 0.75.

If the car were driving to the **right** it would **climb** the hill - we say this is a positive slope. Any time you see the graph of a line that goes up as you move to the right, the slope is **positive**.

If the car kept driving after it reached the top of the hill, it might go down the other side. If the car is driving to the **right** and **descending**, then we would say that the slope is **negative**.



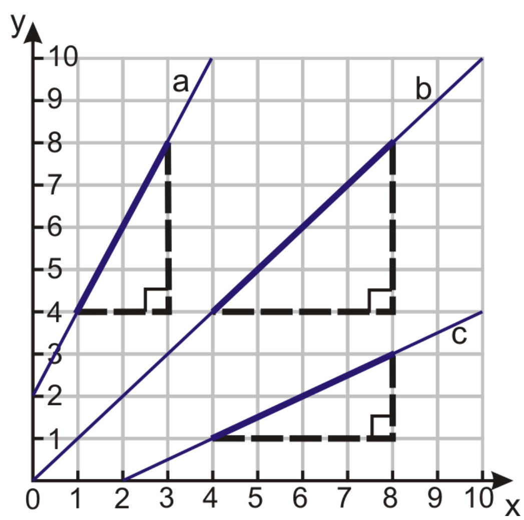
Heres where it gets tricky: If the car turned around instead and drove back down the left side of the hill, the slope of that side would still be positive. This is because the rise would be -3, but the run would be -4 (think of the $x-$axis - if you move from right to left you are moving in the **negative** $x-$direction). That means our slope ratio would be $\frac{-3}{-4}$, and the negatives cancel out to leave 0.75, the same slope as before. In other words, the slope of a line is the same no matter which direction you travel along it.

Find the Slope of a Line

A simple way to find a value for the slope of a line is to draw a right triangle whose hypotenuse runs along the line. Then we just need to measure the distances on the triangle that correspond to the rise (the vertical dimension) and the run (the horizontal dimension).

Example 1

Find the slopes for the three graphs shown.



Solution

There are already right triangles drawn for each of the lines - in future problems youll do this part yourself. Note that it is easiest to make triangles whose vertices are **lattice points** (i.e. points whose coordinates are all integers).

a) The rise shown in this triangle is 4 units; the run is 2 units. The slope is $\frac{4}{2} = 2$.

b) The rise shown in this triangle is 4 units, and the run is also 4 units. The slope is $\frac{4}{4} = 1$.

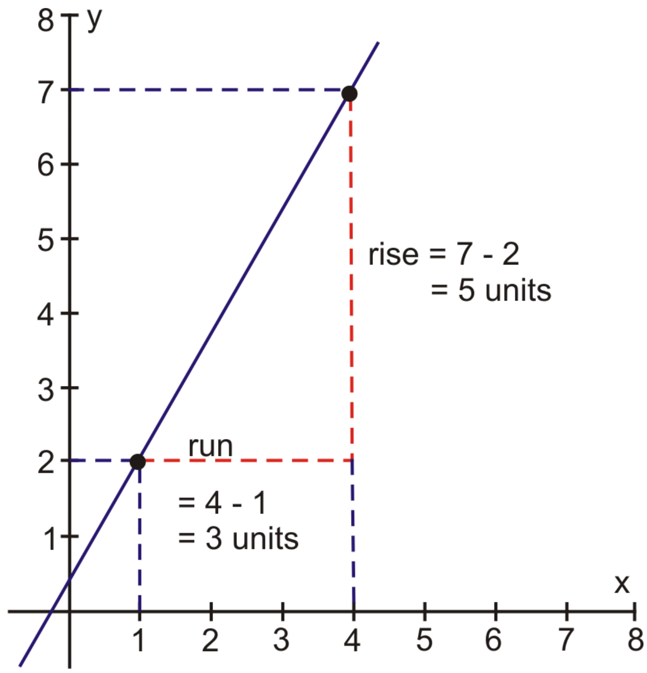
c) The rise shown in this triangle is 2 units, and the run is 4 units. The slope is $\frac{2}{4} = \frac{1}{2}$.

Example 2

Find the slope of the line that passes through the points (1, 2) and (4, 7).

Solution

We already know how to graph a line if were given two points: we simply plot the points and connect them with a line. Heres the graph:



Since we already have coordinates for the vertices of our right triangle, we can quickly work out that the rise is $7-2=5$ and the run is $4-1=3$ (see diagram). So the slope is $\frac{7-2}{4-1}=\frac{5}{3}$.

If you look again at the calculations for the slope, youll notice that the 7 and 2 are the $y-$coordinates of the two points and the 4 and 1 are the $x-$coordinates. This suggests a pattern we can follow to get a general formula for the slope between two points (x_1,y_1) and (x_2,y_2):

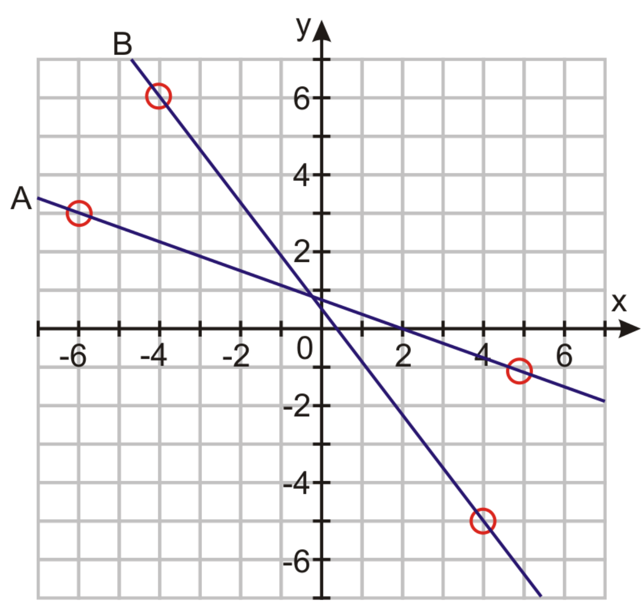
Slope between (x_1,y_1) and $(x_2,y_2)=\frac{y_2-y_1}{x_2-x_1}$

or $m=\frac{\Delta y}{\Delta x}$

In the second equation the letter m denotes the slope (this is a mathematical convention youll see often) and the Greek letter delta (Δ) means ***change***. So another way to express slope is *change in y* divided by *change in x*. In the third section, youll see that it doesnt matter which point you choose as point 1 and which you choose as point 2.

Example 3

Find the slopes of the lines on the graph below.



Solution

Look at the lines - they both slant down (or decrease) as we move from left to right. Both these lines have **negative slope.**

The lines dont pass through very many convenient lattice points, but by looking carefully you can see a few points that look to have integer coordinates. These points have been circled on the graph, and well use them to determine the slope. Well also do our calculations twice, to show that we get the same slope whichever way we choose point 1 and point 2.

For Line A:

$$(x_1, y_1) = (-6, 3) \qquad (x_2, y_2) = (5, -1) \qquad\qquad (x_1, y_1) = (5, -1) \qquad (x_2, y_2) = (-6, 3)$$

$$amp;m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-1) - (3)}{(5) - (-6)} = \frac{-4}{11} \approx -0.364 \qquad m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(3) - (-1)}{(-6) - (5)} = \frac{4}{-11} \approx -0.364$$

For Line B

$$(x_1, y_1) = (-4, 6) \qquad (x_2, y_2) = (4, -5) \qquad\qquad (x_1, y_1) = (4, -5) \qquad (x_2, y_2) = (-4, 6)$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-5) - (6)}{(4) - (-4)} = \frac{-11}{8} = -1.375 \qquad m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(6) - (-5)}{(-4) - (4)} = \frac{11}{-8} = -1.375$$

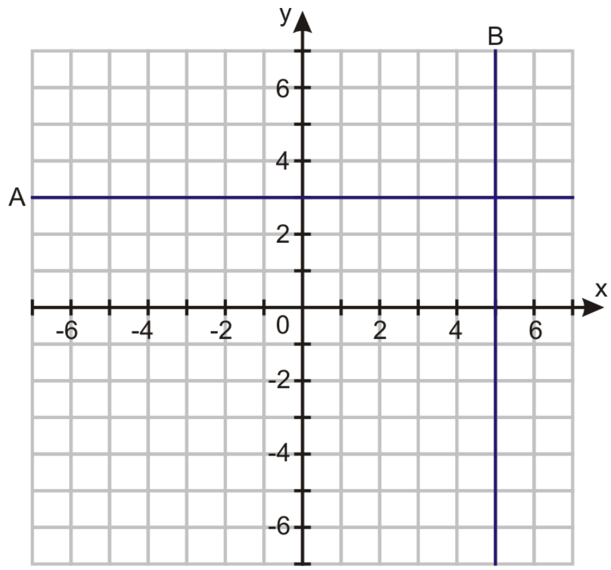
You can see that whichever way round you pick the points, the answers are the same. Either way, **Line** A **has slope -0.364, and Line** B **has slope -1.375.**

Khan Academy has a series of videos on finding the slope of a line, starting at **http://tinyurl.com/7jklqx7**.

Find the Slopes of Horizontal and Vertical lines

Example 4

Determine the slopes of the two lines on the graph below.



Solution

There are 2 lines on the graph: $A(y=3)$ and $B(x=5)$.

Lets pick 2 points on line Asay, $(x_1,y_1)=(-4,3)$ and $(x_2,y_2)=(5,3)$and use our equation for slope:

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{(3)-(3)}{(5)-(-4)}=\frac{0}{9}=0.$$

If you think about it, this makes sense - if y doesnt change as x increases then there is no slope, or rather, the slope is zero. You can see that this must be true for all horizontal lines.

Horizontal lines ($y = constant$) all have a slope of 0.

Now lets consider line B. If we pick the points $(x_1,y_1)=(5,-3)$ and $(x_2,y_2)=(5,4)$, our slope equation is $m=\frac{y_2-y_1}{x_2-x_1}=\frac{(4)-(-3)}{(5)-(5)}=\frac{7}{0}$. But dividing by zero isnt allowed!

In math we often say that a term which involves division by zero is **undefined.** (Technically, the answer can also be said to be infinitely largeor infinitely small, depending on the problem.)

Vertical lines ($x=constant$) all have an infinite (or undefined) slope.

Find a Rate of Change

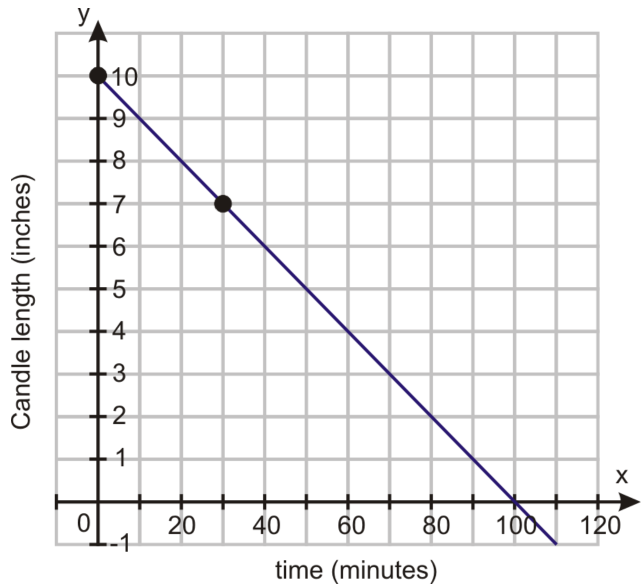
The slope of a function that describes real, measurable quantities is often called a **rate of change**. In that case the slope refers to a change in one quantity (y)**per** unit change in another quantity (x). (This is where the equation $m=\frac{\Delta y}{\Delta x}$ comes inremember that Δy and Δx represent the change in y and x respectively.)

Example 5

A candle has a starting length of 10 inches. 30 minutes after lighting it, the length is 7 inches. Determine the rate of change in length of the candle as it burns. Determine how long the candle takes to completely burn to nothing.

Solution

First well graph the function to visualize what is happening. We have 2 points to start with: we know that at the moment the candle is lit ($time=0$) the length of the candle is 10 inches, and after 30 minutes ($time=30$) the length is 7 inches. Since the candle length depends on the time, well plot time on the horizontal axis, and candle length on the vertical axis.



The rate of change of the candles length is simply the slope of the line. Since we have our 2 points $(x_1, y_1) = (0, 10)$ and $(x_2, y_2) = (30, 7)$, we can use the familiar version of the slope formula:

$$\text{Rate of change} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(7 \text{ inches}) - (10 \text{ inches})}{(30 \text{ minutes}) - (0 \text{ minutes})} = \frac{-3 \text{ inches}}{30 \text{ minutes}} = -0.1 \text{ inches per minute}$$

Note that the slope is negative. A negative rate of change means that the quantity is decreasing with timejust as we would expect the length of a burning candle to do.

To find the point when the candle reaches zero length, we can simply read the $x-$intercept off the graph (100 minutes). We can use the rate equation to verify this algebraically:

$$\begin{aligned}\text{Length burned} &= \text{rate} \times \text{time}\\ 10 &= 0.1 \times 100\end{aligned}$$

Since the candle length was originally 10 inches, our equation confirms that 100 minutes is the time taken.

Example 6

The population of fish in a certain lake increased from 370 to 420 over the months of March and April. At what rate is the population increasing?

Solution

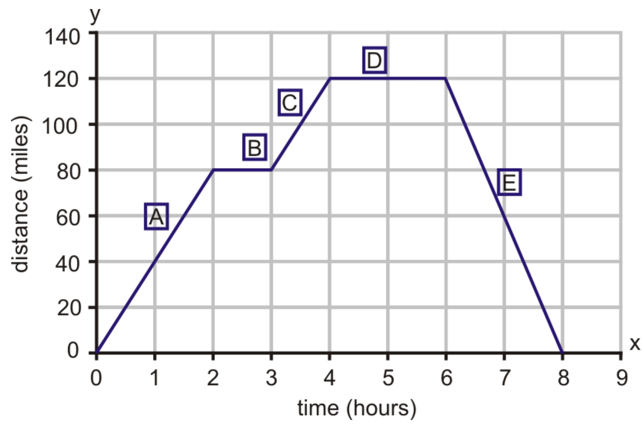
Here we dont have two points from which we can get $x-$ and $y-$coordinates for the slope formula. Instead, well need to use the alternate formula, $m = \frac{\Delta y}{\Delta x}$.

The change in $y-$values, or Δy, is the change in the number of fish, which is $420 - 370 = 50$. The change in $x-$values, Δx, is the amount of time over which this change took place: two months. So $\frac{\Delta y}{\Delta x} = \frac{50 \text{ fish}}{2 \text{ months}}$, or **25 fish per month.**

Interpret a Graph to Compare Rates of Change

Example 7

The graph below represents a trip made by a large delivery truck on a particular day. During the day the truck made two deliveries, one taking an hour and the other taking two hours. Identify what is happening at each stage of the trip (stages A through E).



Solution

Here are the stages of the trip:

a) The truck sets off and travels 80 miles in 2 hours.

b) The truck covers no distance for 2 hours.

c) The truck covers $(120-80) = 40$ miles in 1 hour.

d) The truck covers no distance for 1 hour.

e) The truck covers -120 miles in 2 hours.

Lets look at each section more closely.

A. Rate of change $= \frac{\Delta y}{\Delta x} = \frac{80 \text{ miles}}{2 \text{ hours}} = 40$ miles per hour

Notice that the rate of change is a **speed**or rather, a **velocity.** (The difference between the two is that velocity has a direction, and speed does not. In other words, velocity can be either positive or negative, with negative velocity representing travel in the opposite direction. Youll see the difference more clearly in part E.)

Since velocity equals distance divided by time, the slope (or rate of change) of a distance-time graph is always a velocity.

So during the first part of the trip, the truck travels at a constant speed of 40 mph for 2 hours, covering a distance of 80 miles.

B. The slope here is 0, so the rate of change is 0 mph. The truck is stationary for one hour. This is the first delivery stop.

C. Rate of change $= \frac{\Delta y}{\Delta x} = \frac{(120-80) \text{ miles}}{(4-3) \text{ hours}} = 40$ miles per hour. The truck is traveling at 40 mph.

D. Once again the slope is 0, so the rate of change is 0 mph. The truck is stationary for two hours. This is the second delivery stop. At this point the truck is 120 miles from the start position.

E. Rate of change $= \frac{\Delta y}{\Delta x} = \frac{(0-120) \text{ miles}}{(8-6) \text{ hours}} = \frac{-120 \text{ miles}}{2 \text{ hours}} = -60$ miles per hour. The truck is traveling at *negative* 60 mph.

Wait a negative speed? Does that mean that the truck is reversing? Well, probably not. Its actually the *velocity* and not the speed that is negative, and a negative velocity simply means that the distance *from the starting position* is decreasing with time. The truck is driving in the opposite direction back to where it started from. Since it no longer has 2 heavy loads, it travels faster (60 mph instead of 40 mph), covering the 120 mile return trip in 2 hours. Its *speed*

is 60 mph, and its *velocity* is -60 mph, because it is traveling in the opposite direction from when it started out.

Lesson Summary

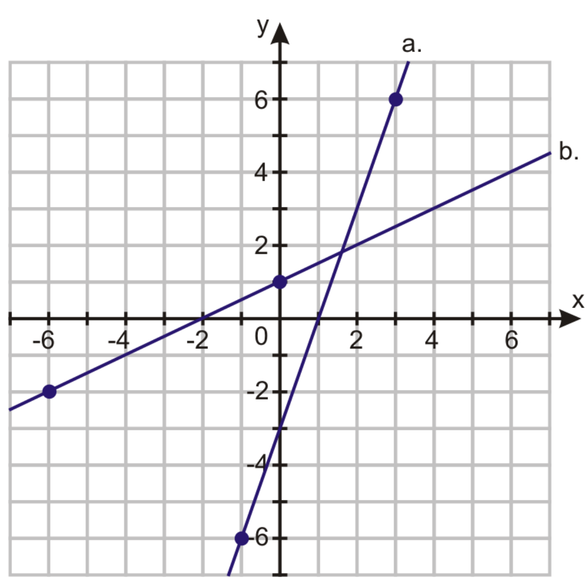
- **Slope** is a measure of change in the vertical direction for each step in the horizontal direction. Slope is often represented as m.
- Slope can be expressed as $\frac{\text{rise}}{\text{run}}$, or $\frac{\Delta y}{\Delta x}$.
- The slope between two points (x_1, y_1) and (x_2, y_2) is equal to $\frac{y_2 - y_1}{x_2 - x_1}$.
- **Horizontal lines** (where $y = a$ constant) all have a slope of 0.
- **Vertical lines** (where $x = a$ constant) all have an infinite (or undefined) slope.
- The slope (or **rate of change**) of a distance-time graph is a **velocity.**

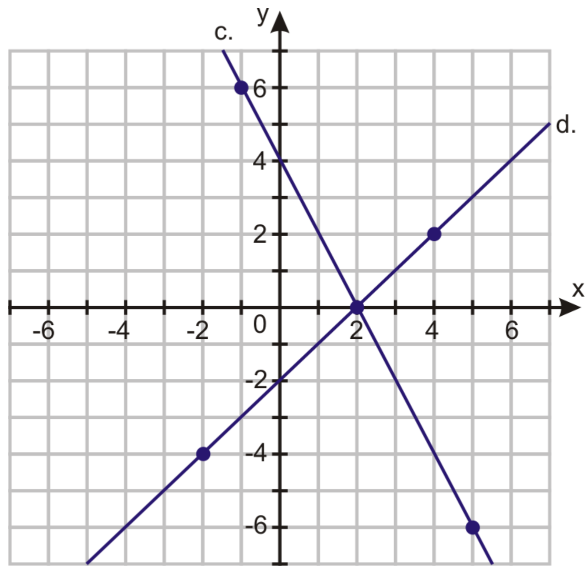
Practice Set

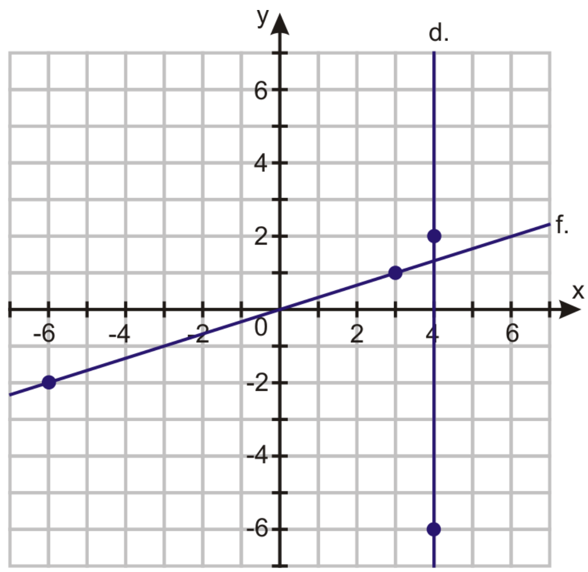
1. Use the slope formula to find the slope of the line that passes through each pair of points.

   (a) (-5, 7) and (0, 0)
   (b) (-3, -5) and (3, 11)
   (c) (3, -5) and (-2, 9)
   (d) (-5, 7) and (-5, 11)
   (e) (9, 9) and (-9, -9)
   (f) (3, 5) and (-2, 7)
   (g) (2.5, 3) and (8, 3.5)

2. For each line in the graphs below, use the points indicated to determine the slope.







3. For each line in the graphs above, imagine another line with the same slope that passes through the point (1, 1), and name one more point on that line.

10.2 Graphs Using Slope-Intercept Form

Learning Objectives

- Identify the slope and $y-$intercept of equations and graphs.
- Graph an equation in slope-intercept form.
- Understand what happens when you change the slope or intercept of a line.
- Identify parallel lines from their equations.

Introduction

The total profit of a business is described by the equation $y = 15000x - 80000$, where x is the number of months the business has been running. How much profit is the business making per month, and what were its start-up costs? How much profit will it have made in a year?

Identify Slope and intercept

So far, weve been writing a lot of our equations in **slope-intercept form**that is, weve been writing them in the form $y = mx + b$, where m and b are both constants. It just so happens that m is the slope and the point $(0, b)$ is the $y-$intercept of the graph of the equation, which gives us enough information to draw the graph quickly.

Example 1

Identify the slope and $y-$intercept of the following equations.

a) $y = 3x + 2$

b) $y = 0.5x - 3$

c) $y = -7x$

d) $y = -4$

Solution

a) Comparing

slope

$$y = 3x + 2 \text{ with } y = mx + b$$

intercept

, we can see that $m = 3$ and $b = 2$. So $y = 3x + 2$ has a **slope of 3** and a $y-$**intercept of (0, 2).**

b)

slope

$$y = 0.5x - 3$$

intercept

has a **slope of 0.5** and a $y-$**intercept of (0, -3).**

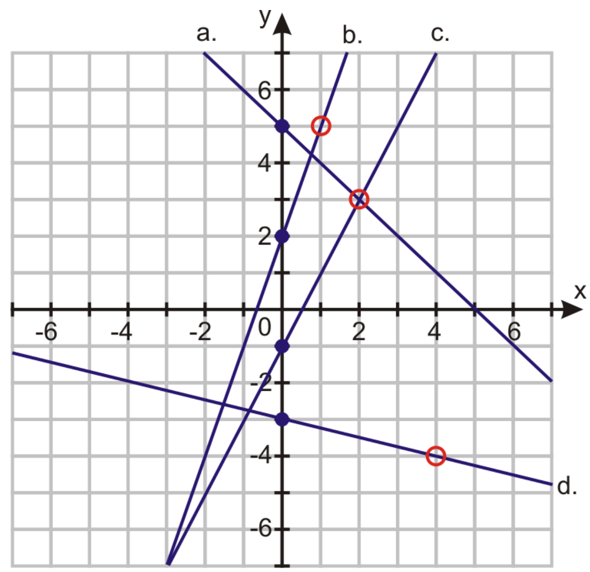
Notice that the intercept is **negative**. The $b-$term includes the sign of the operator (plus or minus) in front of the numberfor example, $y = 0.5x - 3$ is identical to $y = 0.5x + (-3)$, and that means that b is -3, not just 3.

c) At first glance, this equation doesnt look like its in slope-intercept form. But we can rewrite it as $y = -7x + 0$, and that means it has a **slope of -7** and a $y-$**intercept of (0, 0).** Notice that the slope is negative and the line passes through the origin.

d) We can rewrite this one as $y = 0x - 4$, giving us a **slope of 0** and a $y-$**intercept of (0, -4).** This is a horizontal line.

Example 2

Identify the slope and $y-$intercept of the lines on the graph shown below.



The intercepts have been marked, as well as some convenient lattice points that the lines pass through.

Solution

a) **The $y-$intercept is (0, 5).** The line also passes through (2, 3), so the slope is $\frac{\Delta y}{\Delta x} = \frac{-2}{2} = -1$.

b) **The $y-$intercept is (0, 2).** The line also passes through (1, 5), so the slope is $\frac{\Delta y}{\Delta x} = \frac{3}{1} = 3$.

c) **The $y-$intercept is (0, -1).** The line also passes through (2, 3), so the slope is $\frac{\Delta y}{\Delta x} = \frac{4}{2} = 2$.

d) **The $y-$intercept is (0, -3).** The line also passes through (4, -4), so the slope is $\frac{\Delta y}{\Delta x} = \frac{-1}{4} = -\frac{1}{4}$ or -0.25.

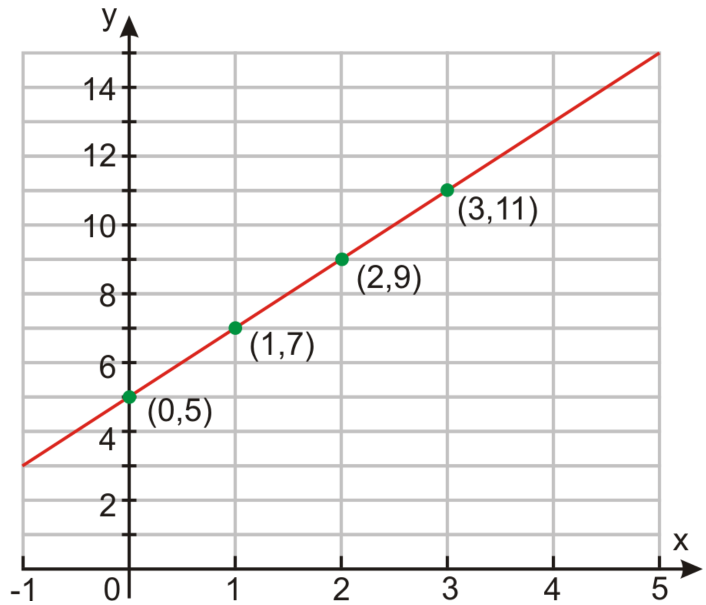
Graph an Equation in Slope-Intercept Form

Once we know the slope and intercept of a line, its easy to graph it. Just remember what slope means. Let's look back at this example from a previous lesson.

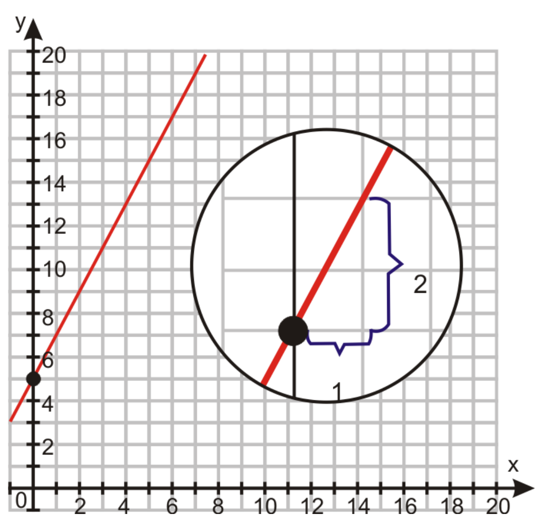
Ali is trying to work out a trick that his friend showed him. His friend started by asking him to think of a number, then double it, then add five to the result. Ali has written down a rule to describe the first part of the trick. He is using the letter x to stand for the number he thought of and the letter y to represent the final result of applying the rule. He wrote his rule in the form of an equation: $y = 2x + 5$.

Help him visualize what is going on by graphing the function that this rule describes.

In that example, we constructed a table of values, and used that table to plot some points to create our graph.



We also saw another way to graph this equation. Just by looking at the equation, we could see that the $y-$intercept was (0, 5), so we could start by plotting that point. Then we could also see that the slope was 2, so we could find another point on the graph by going over 1 unit and up 2 units. The graph would then be the line between those two points.



Heres another problem where we can use the same method.

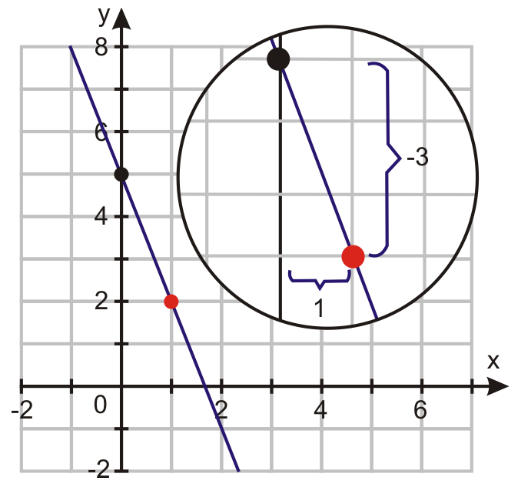
Example 3

Graph the following function: $y = -3x + 5$

Solution

To graph the function without making a table, follow these steps:

1. Identify the $y-$intercept: $b = 5$
2. Plot the intercept: (0, 5)
3. Identify the slope: $m = -3$. (This is equal to $\frac{-3}{1}$, so the **rise** is -3 and the **run** is 1.)
4. Move **over** 1 unit and **down** 3 units to find another point on the line: (1, 2)
5. Draw the line through the points (0, 5) and (1, 2).



Notice that to graph this equation based on its slope, we had to find the rise and runand it was easiest to do that when the slope was expressed as a fraction. Thats true in general: to graph a line with a particular slope, its easiest to first express the slope as a fraction in simplest form, and then read off the numerator and the denominator of the fraction to get the rise and run of the graph.

Example 4

*Find integer values for the **rise** and **run** of the following slopes, then graph lines with corresponding slopes.*

a) $m = 3$

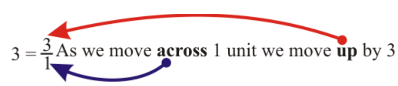
b) $m = -2$

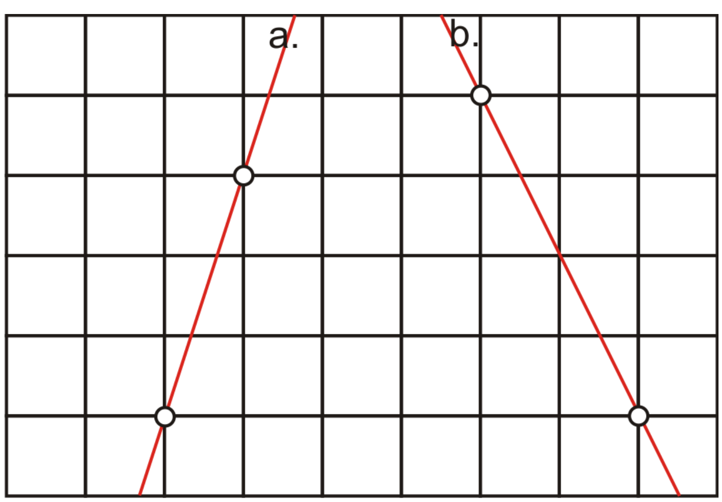
c) $m = 0.75$

d) $m = -0.375$

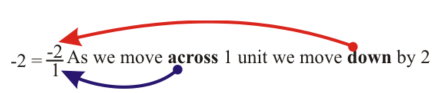
Solution

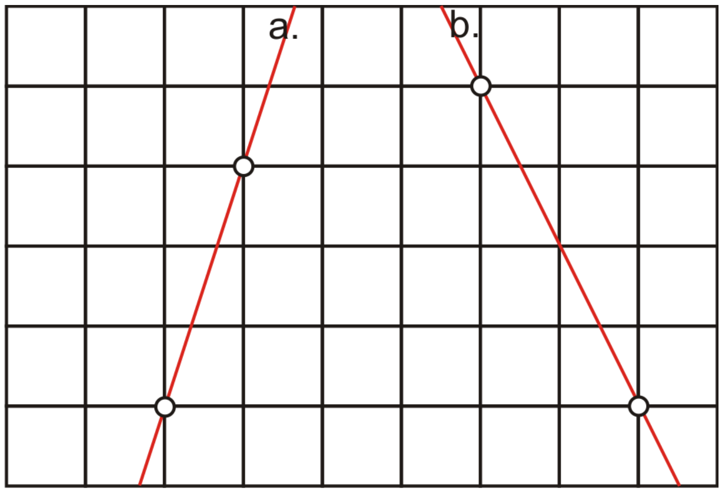
a)



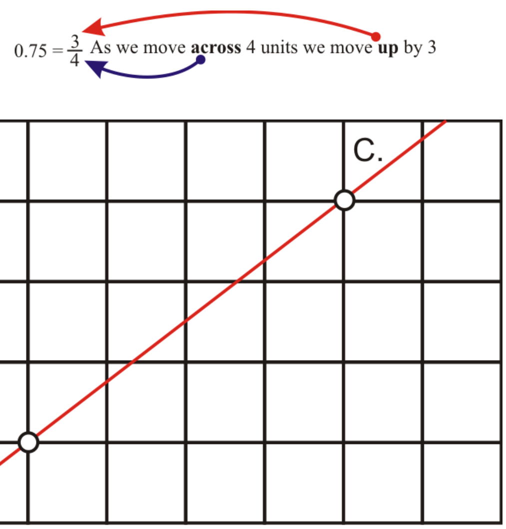


b)

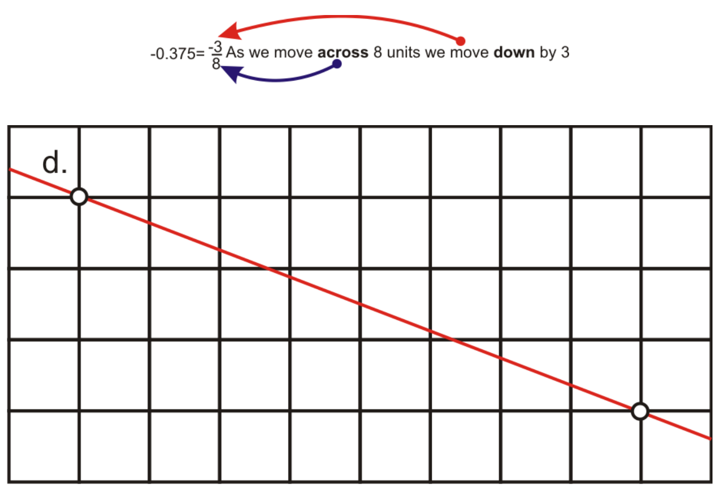




c)

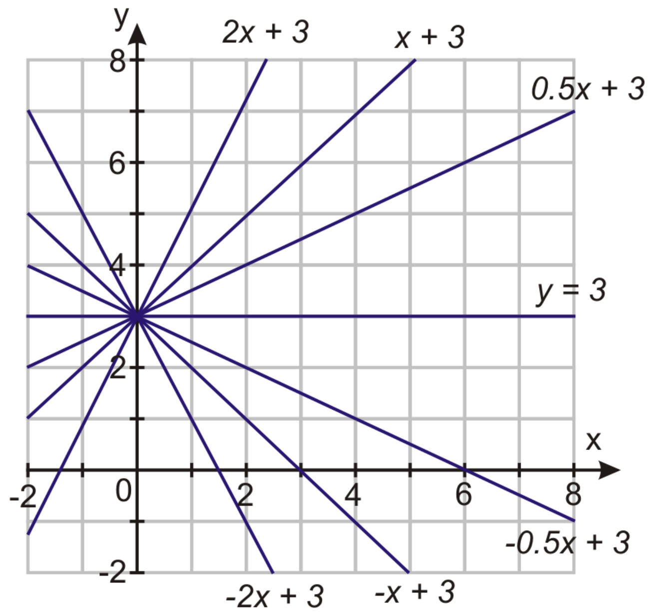


d)



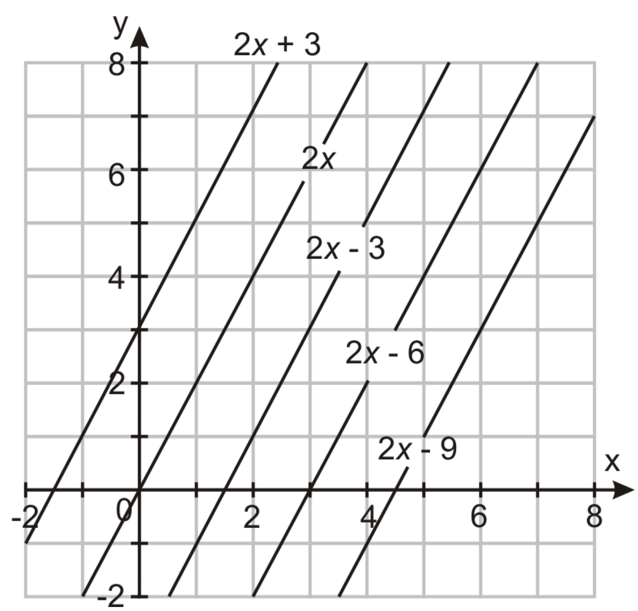
Changing the Slope or Intercept of a Line

The following graph shows a number of lines with different slopes, but all with the same $y-$intercept: (0, 3).



You can see that all the functions with positive slopes increase as we move from left to right, while all functions with negative slopes decrease as we move from left to right. Another thing to notice is that the greater the slope, the steeper the graph.

This graph shows a number of lines with the same slope, but different $y-$intercepts.



Notice that changing the intercept simply translates (shifts) the graph up or down. Take a point on the graph of $y = 2x$, such as (1, 2). The corresponding point on $y = 2x + 3$ would be (1, 5). Adding 3 to the $y-$intercept means we also add 3 to every other $y-$value on the graph. Similarly, the corresponding point on the $y = 2x - 3$ line would be (1, -1); we would subtract 3 from the $y-$value and from every other $y-$value.

Notice also that these lines all appear to be parallel. Are they truly parallel?

To answer that question, well use a technique that youll learn more about in a later chapter. Well take 2 of the equationssay, $y = 2x$ and $y = 2x + 3$and solve for values of x and y that satisfy both equations. That will tell us at what point those two lines intersect, if any. (Remember that **parallel lines**, by definition, are lines that dont intersect.)

So what values would satisfy both $y = 2x$ and $y = 2x+3$? Well, if both of those equations were true, then y would be equal to both $2x$ and $2x+3$, which means those two expressions would also be equal to each other. So we can get our answer by solving the equation $2x = 2x+3$.

But what happens when we try to solve that equation? If we subtract $2x$ from both sides, we end up with $0 = 3$. That cant be true no matter what x equals. And that means that there just isnt any value for x that will make both of the equations we started out with true. In other words, there isnt any point where those two lines intersect. They are parallel, just as we thought.

And wed find out the same thing no matter which two lines wed chosen. In general, since changing the intercept of a line just results in shifting the graph up or down, the new line will always be parallel to the old line as long as the slope stays the same.

Any two lines with identical slopes are parallel.

Further Practice

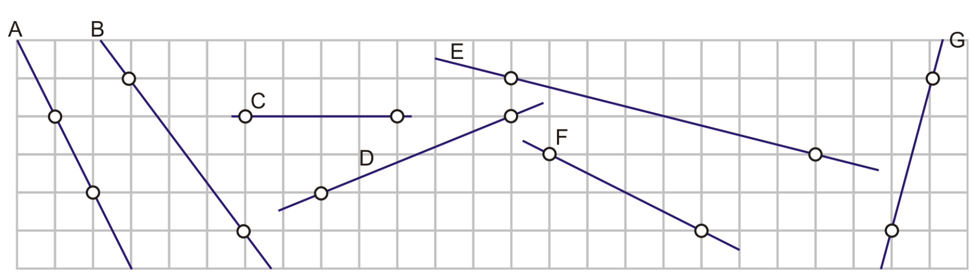
To get a better understanding of what happens when you change the slope or the $y-$intercept of a linear equation, try playing with the Java applet athttp://standards.nctm.org/document/eexamples/chap7/7.5/index.htm. **http://tinyurl.com/7laf6oh.**

Lesson Summary

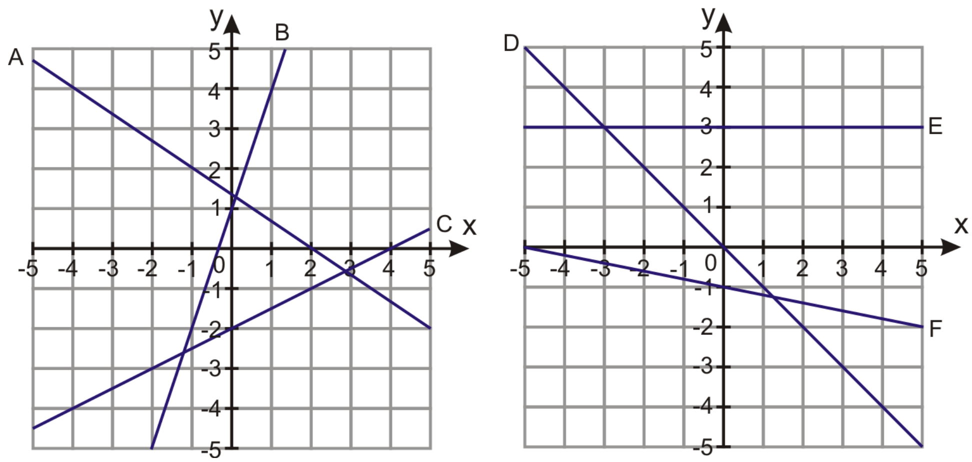
- A common form of a line (linear equation) is **slope-intercept form:** $y = mx+b$, where m is the slope and the point $(0,b)$ is the $y-$intercept
- Graphing a line in slope-intercept form is a matter of first plotting the $y-$intercept $(0,b)$, then finding a second point based on the slope, and using those two points to graph the line.
- Any two lines with identical slopes are **parallel.**

Practice Set

1. Identify the slope and $y-$intercept for the following equations.
   (a) $y = 2x+5$
   (b) $y = -0.2x+7$
   (c) $y = x$
   (d) $y = 3.75$
2. Identify the slope of the following lines.



3. Identify the slope and $y-$intercept for the following functions.



4. Plot the following functions on a graph.
   (a) $y = 2x + 5$
   (b) $y = -0.2x + 7$
   (c) $y = x$
   (d) $y = 3.75$
5. Which two of the following lines are parallel?
   (a) $y = 2x + 5$
   (b) $y = -0.2x + 7$
   (c) $y = x$
   (d) $y = 3.75$
   (e) $y = -\frac{1}{5}x - 11$
   (f) $y = -5x + 5$
   (g) $y = -3x + 11$
   (h) $y = 3x + 3.5$
6. What is the $y-$intercept of the line passing through (1, -4) and (3, 2)?
7. What is the $y-$intercept of the line with slope -2 that passes through (3, 1)?

10.3 Problem-Solving Strategies - Creating and Interpreting Graphs

Introduction

In this chapter, weve been solving problems where quantities are linearly related to each other. In this section, well look at a few examples of linear relationships that occur in real-world problems, and see how we can solve them using graphs. Remember back to our Problem Solving Plan:

1. **Understand the Problem**
2. **Devise a PlanTranslate**
3. **Carry Out the PlanSolve**
4. **LookCheck and Interpret**

Example 1

A cell phone company is offering its costumers the following deal: You can buy a new cell phone for $60 and pay a monthly flat rate of $40 per month for unlimited calls. How much money will this deal cost you after 9 months?

Solution

Lets follow the problem solving plan.

Step 1: The phone costs $60; the calling plan costs $40 per month.

Let $x =$ number of months.

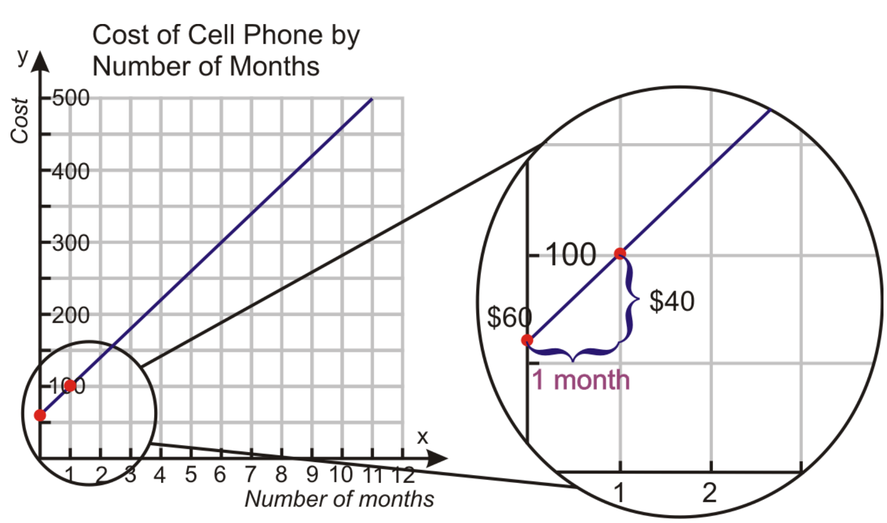
Let $y =$ total cost.

Step 2: We can solve this problem by making a graph that shows the number of months on the horizontal axis and the cost on the vertical axis.

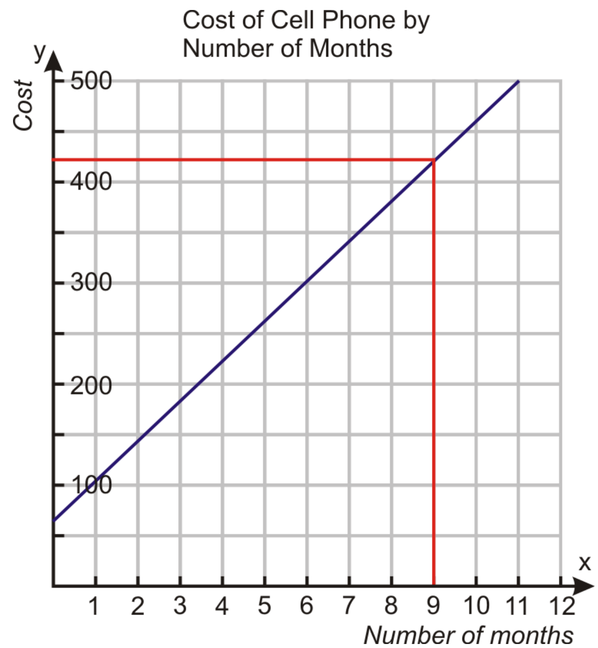
Since you pay $60 when you get the phone, the $y-$intercept is (0, 60).

You pay $40 for each month, so the cost rises by $40 for 1 month, so the slope is 40.

We can use this information to graph the situation.



Step 3: The question was How much will this deal cost after 9 months? We can now read the answer from the graph. We draw a vertical line from 9 months until it meets the graph, and then draw a horizontal line until it meets the vertical axis.



We see that after 9 months **you pay approximately $420.**

Step 4: To check if this is correct, lets think of the deal again.

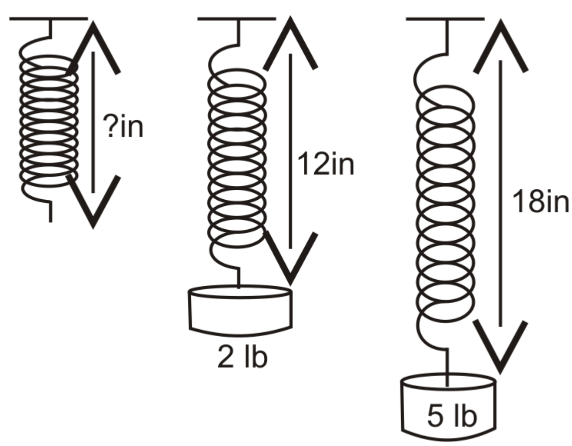
Originally, you pay $60 and then $40 a month for 9 months.

$$\begin{aligned}\text{Phone} &= \$60\\ \text{Calling plan} &= \$40 \times 9 = \$360\\ \text{Total cost} &= \$420.\end{aligned}$$

The answer checks out.

Example 2

A stretched spring has a length of 12 inches when a weight of 2 lbs is attached to the spring. The same spring has a length of 18 inches when a weight of 5 lbs is attached to the spring. What is the length of the spring when no weights are attached?



Solution

Step 1: We know: the length of the spring = 12 inches when weight = 2 lbs

the length of the spring = 18 inches when weight = 5 lbs

We want: the length of the spring when weight = 0 lbs

Let $x =$ the weight attached to the spring.

Let $y =$ the length of the spring.

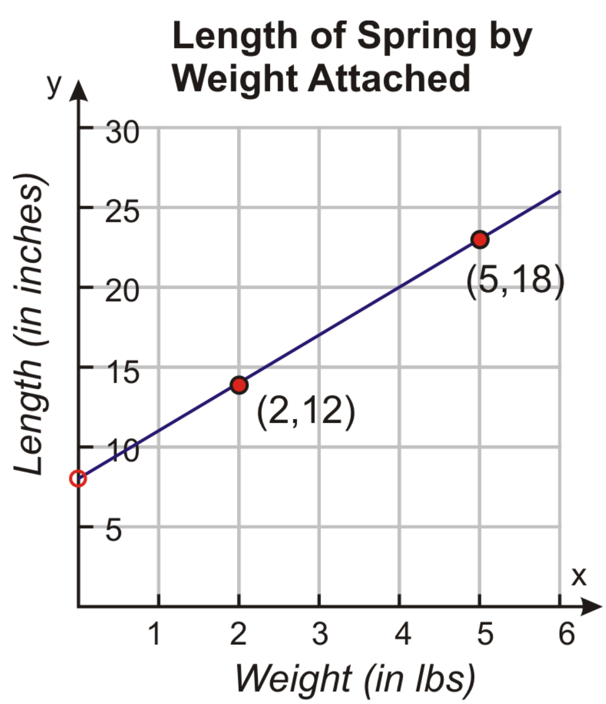
Step 2: We can solve this problem by making a graph that shows the weight on the horizontal axis and the length of the spring on the vertical axis.

We have two points we can graph:

When the weight is 2 lbs, the length of the spring is 12 inches. This gives point (2, 12).

When the weight is 5 lbs, the length of the spring is 18 inches. This gives point (5, 18).

Graphing those two points and connecting them gives us our line.



Step 3: The question was: *What is the length of the spring when no weights are attached?*

We can answer this question by reading the graph we just made. When there is no weight on the spring, the $x-$value equals zero, so we are just looking for the $y-$intercept of the graph. On the graph, the $y-$intercept appears to be approximately 8 inches.

Step 4: To check if this correct, lets think of the problem again.

You can see that the length of the spring goes up by 6 inches when the weight is increased by 3 lbs.

To find the length of the spring when there is no weight attached, we can look at the spring when there are 2 lbs attached. For each pound we take off, the spring will shorten by 2 inches. If we take off 2 lbs, the spring will be shorter by 4 inches. So, the length of the spring with no weights is 12 inches - 4 inches = 8 inches.

The answer checks out.

Example 3

Christine took 1 hour to read 22 pages of Harry Potter. *She has 100 pages left to read in order to finish the book. How much time should she expect to spend reading in order to finish the book?*

Solution

Step 1: We know - Christine takes 1 hour to read 22 pages.

We want - How much time it takes to read 100 pages.

Let $x =$ the time expressed in hours.

Let $y =$ the number of pages.

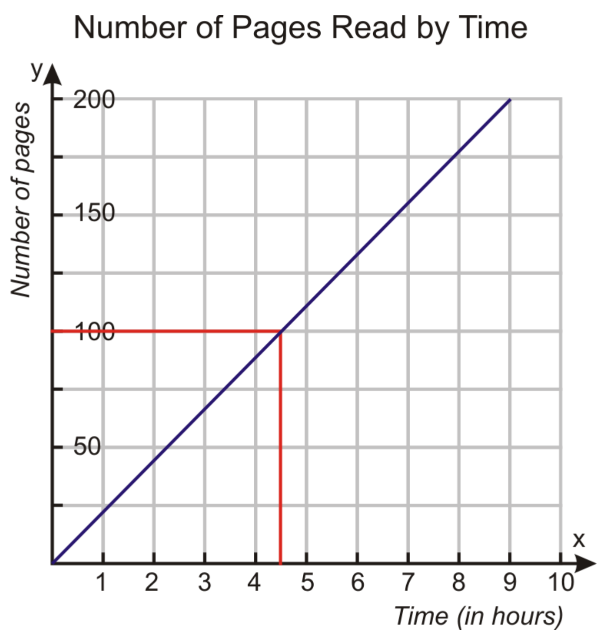
Step 2: We can solve this problem by making a graph that shows the number of hours spent reading on the horizontal axis and the number of pages on the vertical axis.

We have two points we can graph:

Christine takes 1 hour to read 22 pages. This gives point (1, 22).

A second point is not given, but we know that Christine would take 0 hours to read 0 pages. This gives point (0, 0).

Graphing those two points and connecting them gives us our line.



Step 3: The question was: How much time should Christine expect to spend reading 100 pages? We can find the answer from reading the graph - we draw a horizontal line from 100 pages until it meets the graph and then we draw the vertical until it meets the horizontal axis. We see that it takes **approximately 4.5 hours** to read the remaining 100 pages.

Step 4: To check if this correct, lets think of the problem again.

We know that Christine reads 22 pages per hour - this is the rate at which she is reading. To find how many hours it takes her to read 100 pages, we divide the number of pages by the rate. In this case, $\frac{100 \ pages}{22 \ pages/hour} = 4.54 \ hours$. This is very close to the answer we got from reading the graph.

The answer checks out.

Example 4

Aatif wants to buy a surfboard that costs \$249. He was given a birthday present of \$50 and he has a summer job that pays him \$6.50 per hour. To be able to buy the surfboard, how many hours does he need to work?

Solution

Step 1: We know - The surfboard costs \$249.

Aatif has \$50.

His job pays \$6.50 per hour.

We want - How many hours Aatif needs to work to buy the surfboard.

Let $x =$ the time expressed in hours

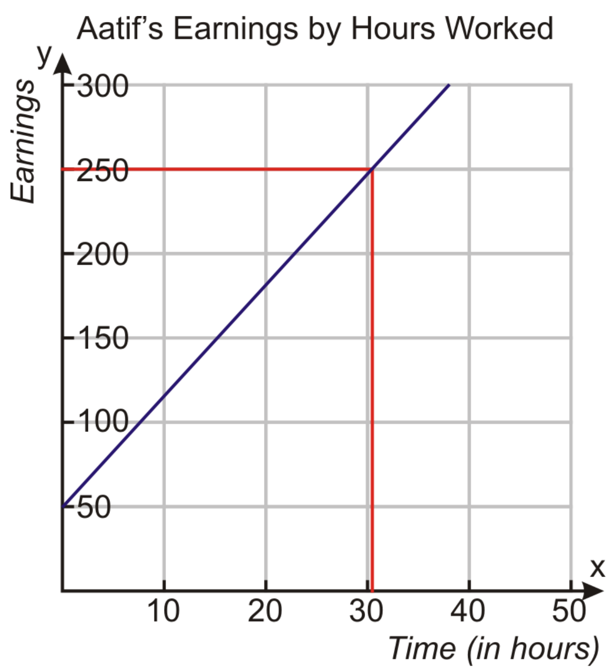
Let $y =$ Aatifs earnings

Step 2: We can solve this problem by making a graph that shows the number of hours spent working on the horizontal axis and Aatifs earnings on the vertical axis.

Aatif has \$50 at the beginning. This is the $y-$intercept: (0, 50).

He earns \$6.50 per hour.

We graph the $y-$intercept of (0, 50), and we know that for each unit in the horizontal direction, the line rises by 6.5 units in the vertical direction. Here is the line that describes this situation.



Step 3: The question was: *How many hours does Aatif need to work to buy the surfboard?*

We find the answer from reading the graph - since the surfboard costs \$249, we draw a horizontal line from \$249 on the vertical axis until it meets the graph and then we draw a vertical line downwards until it meets the horizontal axis. We see that it takes **approximately 31 hours** to earn the money.

Step 4: To check if this correct, lets think of the problem again.

We know that Aatif has \$50 and needs \$249 to buy the surfboard. So, he needs to earn $\$249 - \$50 = \$199$ from his job.

His job pays \$6.50 per hour. To find how many hours he need to work we divide: $\frac{\$199}{\$6.50/hour} = 30.6\ hours$. This is very close to the answer we got from reading the graph.

The answer checks out.

Lesson Summary

The four steps of the **problem solving plan** when using graphs are:

1. **Understand the Problem**
2. **Devise a PlanTranslate:** Make a graph.
3. **Carry Out the PlanSolve:** Use the graph to answer the question asked.
4. **LookCheck and Interpret**

Practice Set

Solve the following problems by making a graph and reading it.

1. A gym is offering a deal to new members. Customers can sign up by paying a registration fee of \$200 and a monthly fee of \$39.
   (a) How much will this membership cost a member by the end of the year?
   (b) The old membership rate was \$49 a month with a registration fee of \$100. How much more would a years membership cost at that rate?
   (c) **Bonus:** For what number of months would the two membership rates be the same?
2. A candle is burning at a linear rate. The candle measures five inches two minutes after it was lit. It measures three inches eight minutes after it was lit.
   (a) What was the original length of the candle?
   (b) How long will it take to burn down to a half-inch stub?
   (c) Six half-inch stubs of candle can be melted together to make a new candle measuring $2\frac{5}{6}$ inches (a little wax gets lost in the process). How many stubs would it take to make three candles the size of the original candle?
3. A dipped candle is made by taking a wick and dipping it repeatedly in melted wax. The candle gets a little bit thicker with each added layer of wax. After it has been dipped three times, the candle is 6.5 mm thick. After it has been dipped six times, it is 11 mm thick.
   (a) How thick is the wick before the wax is added?
   (b) How many times does the wick need to be dipped to create a candle 2 cm thick?
4. Tali is trying to find the thickness of a page of his telephone book. In order to do this, he takes a measurement and finds out that 55 pages measures $\frac{1}{8}$ *inch*. What is the thickness of one page of the phone book?
5. Bobby and Petra are running a lemonade stand and they charge 45 cents for each glass of lemonade. In order to break even they must make \$25.
   (a) How many glasses of lemonade must they sell to break even?
   (b) When theyve sold \$18 worth of lemonade, they realize that they only have enough lemons left to make 10 more glasses. To break even now, theyll need to sell those last 10 glasses at a higher price. What does the new price need to be?
6. Dale is making cookies using a recipe that calls for 2.5 cups of flour for two dozen cookies. How many cups of flour does he need to make five dozen cookies?
7. To buy a car, Jason makes a down payment of \$1500 and pays \$350 per month in installments.
   (a) How much money has Jason paid at the end of one year?
   (b) If the total cost of the car is \$8500, how long will it take Jason to finish paying it off?
   (c) The resale value of the car decreases by \$100 each month from the original purchase price. If Jason sells the car as soon as he finishes paying it off, how much will he get for it?

8. Anne transplants a rose seedling in her garden. She wants to track the growth of the rose so she measures its height every week. On the third week, she finds that the rose is 10 inches tall and on the eleventh week she finds that the rose is 14 inches tall. Assuming the rose grows linearly with time, what was the height of the rose when Anne planted it?
9. Ravi hangs from a giant spring whose length is 5 m. When his child Nimi hangs from the spring its length is 2 m. Ravi weighs 160 lbs and Nimi weighs 40 lbs. Write the equation for this problem in slope-intercept form. What should we expect the length of the spring to be when his wife Amardeep, who weighs 140 lbs, hangs from it?
10. Nadia is placing different weights on a spring and measuring the length of the stretched spring. She finds that for a 100 gram weight the length of the stretched spring is 20 cm and for a 300 gram weight the length of the stretched spring is 25 cm.
    (a) What is the unstretched length of the spring?
    (b) If the spring can only stretch to twice its unstretched length before it breaks, how much weight can it hold?
11. Andrew is a submarine commander. He decides to surface his submarine to periscope depth. It takes him 20 minutes to get from a depth of 400 feet to a depth of 50 feet.
    (a) What was the submarines depth five minutes after it started surfacing?
    (b) How much longer would it take at that rate to get all the way to the surface?
12. Kierstas phone has completely run out of battery power when she puts it on the charger. Ten minutes later, when the phone is 40% recharged, Kierstas friend Danielle calls and Kiersta takes the phone off the charger to talk to her. When she hangs up 45 minutes later, her phone has 10% of its charge left. Then she gets another call from her friend Kwan.
    (a) How long can she spend talking to Kwan before the battery runs out again?
    (b) If she puts the phone back on the charger afterward, how long will it take to recharge completely?
13. Marji is painting a 75-foot fence. She starts applying the first coat of paint at 2 PM, and by 2:10 she has painted 30 feet of the fence. At 2:15, her husband, who paints about $\frac{2}{3}$ as fast as she does, comes to join her.
    (a) How much of the fence has Marji painted when her husband joins in?
    (b) When will they have painted the whole fence?
    (c) How long will it take them to apply the second coat of paint if they work together the whole time?

CHAPTER 11 More on Linear Functions

Chapter Outline

11.1 Writing Linear Equations in Slope-Intercept Form

Previously, you learned how to graph solutions to two-variable equations in slope-intercept form. This lesson focuses on how to write an equation for a graphed line. There are two things you will need from the graph to write the equation in slope-intercept form:

1. The $y-$intercept of the graph and
2. The slope of the line.

Having these two things will allow you to make the appropriate substitutions in the slope-intercept formula. Recall from the last chapter,

Slope-intercept form: $y = (slope)x + (y - intercept)$ or $y = mx + b$

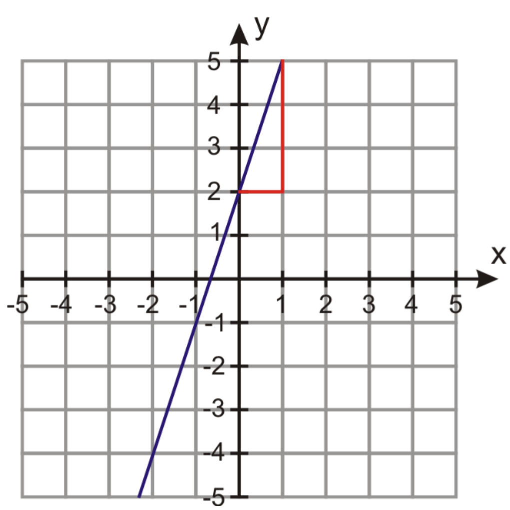
Example 1: Write the equation for a line with a slope of 4 and a $y-$intercept (0, -3).

Solution: Slope-intercept form needs two things: the slope and $y-$intercept. To write the equation, you substitute the values into the formula.

$$\begin{aligned}
y &= (slope)x + (y - intercept) \\
y &= 4x + -3 \\
y &= 4x - 3
\end{aligned}$$

You can also use a graphed line to determine the slope and $y-$intercept.

Example 2: *Use the graph below to write its equation in slope-intercept form.*



Solution: The $y-$intercept is (0, 2). Using the slope triangle, you can determine the slope is $\frac{rise}{run} = \frac{+3}{+1} = \frac{3}{1}$. Substituting the value 2 for b and the value 3 for m, the equation for this line is $y = 3x + 2$.

Writing an Equation Given the Slope and a Point

Sometimes it may be difficult to determine the $y-$intercept. Perhaps the $y-$intercept is rational instead of an integer. Maybe you dont know the $y-$intercept. All you have is the slope and an ordered pair. You can use this information to write the equation in slope-intercept form. To do so, you will need to follow several steps.

Step 1: Begin by writing the formula for slope-intercept form $y = mx + b$.

Step 2: Substitute the given slope for m.

Step 3: Use the ordered pair you are given (x, y), substitute these values for the variables x and y in the equation.

Step 4: Solve for b (the $y-$intercept of the graph).

Step 5: Rewrite the original equation in step 1, substituting the slope for m and the $y-$intercept for b.

Example 3: *Write an equation for a line with slope of 4 that contains the ordered pair (-1, 5).*

Solution:

Step 1: Begin by writing the formula for slope-intercept form.

$$y = mx + b$$

Step 2: Substitute the given slope for m.

$$y = 4x + b$$

Step 3: Use the ordered pair you are given (-1, 5), substitute these values for the variables x and y in the equation.

$$5 = (4)(-1) + b$$

Step 4: Solve for b (the $y-$intercept of the graph).

$$\begin{aligned} 5 &= -4 + b \\ 5 + 4 &= -4 + 4 + b \\ 9 &= b \end{aligned}$$

Step 5: Rewrite $y = mx + b$, substituting the slope for m and the $y-$intercept for b

$$y = 4x + 9$$

Example 4: Write the equation for a line with a slope of -3 containing the point (3, -5).

Solution: Using the five-steps from above:

$$\begin{aligned} y &= (slope)x + (y - intercept) \\ y &= -3x + b \\ -5 &= -3(3) + b \\ -5 &= -9 + b \\ 4 &= b \\ y &= -3x + 4 \end{aligned}$$

Writing an Equation Given Two Points

In many cases, especially real-world situations, you are not given the slope nor are you given the $y-$intercept. You might only have two points to use to determine the equation of the line.

To find an equation for a line between two points, you need two things:

1. The $y-$intercept of the graph; and
2. The slope of the line.

Previously, you learned how to determine the slope between two points. Lets repeat the formula here:

The slope between any two points (x_1, y_1) *and* (x_2, y_2) *is*: $slope = \frac{y_2-y_1}{x_2-x_1}$

The procedure for determining a line given two points is the same five-step process as writing an equation given the slope and a point.

Example 5: *Write the equation for the line containing the points (3, 2) and (-2, 4).*

Solution: You need the slope of the line. Find the line's slope by using the formula. Choose one ordered pair to represent (x_1, y_1) and the other ordered pair to represent (x_2, y_2).

$$slope = \frac{y_2-y_1}{x_2-x_1} = \frac{4-2}{-2-3} = \frac{-2}{5} = -\frac{2}{5}$$

Now use the five-step process to find the equation for this line.

Step 1: Begin by writing the formula for slope-intercept form.

$$y = mx + b$$

Step 2: Substitute the given slope for m.

$$y = -\frac{2}{5}x + b$$

Step 3: Use one of the ordered pairs you are given (-2, 4), substitute these values for the variables x and y in the equation.

$$4 = \left(-\frac{2}{5}\right)(-2) + b$$

Step 4: Solve for b (the $y-$intercept of the graph).

$$\begin{aligned} 4 &= \frac{4}{5} + b \\ 4 - \frac{4}{5} &= \frac{4}{5} - \frac{4}{5} + b \\ \frac{16}{5} &= b \end{aligned}$$

Step 5: Rewrite $y = mx + b$, substituting the slope for m and the $y-$intercept for b.

$$y = -\frac{2}{5}x + \frac{16}{5}$$

Example 6: Write the equation for a line containing the points (-4, 1) and (-2, 3).

Solution:

1. Start with the slopeintercept form of the line $y = mx + b$.
2. Find the slope of the line. $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-1}{-2-(-4)} = \frac{2}{2} = 1$
3. Substitute the value of slope for $m : y = (1)x + b$
4. Substitute the coordinate (-2, 3) into the equation for the variables x and $y : 3 = -2 + b \Rightarrow b = 5$
5. Rewrite the equation, substituting the slope for m and the $y-$intercept for b. $y = x + 5$

Writing a Function in Slope-Intercept Form

Remember that a linear function has the form $f(x) = mx + b$. Here $f(x)$ represents the y values of the equation or the graph. So $y = f(x)$ and they are often used interchangeably. Using the functional notation in an equation often provides you with more information.

For instance, the expression $f(x) = mx + b$ shows clearly that x is the independent variable because you **substitute** values of x into the function and perform a series of operations on the value of x in order to calculate the values of the dependent variable, y.

$$x$$

$$y = f(\ \ ) = m\,\_\_\_ + b$$

In this case when you substitute x into the function, the function tells you to multiply it by m and then add b to the result. This process generates all the values of y you need.

Example 7: Consider the function $f(x) = 3x - 4$. *Find* $f(2), f(0)$, *and* $f(-1)$.

Solution: Each number in parentheses is a value of x that you need to substitute into the equation of the function.

$f(2) = 3(2) - 4 = 6 - 4 = 2$ $\quad$ $f(0) = 3(0) - 4 = 0 - 4 = -4$ $\quad$ $f(-1) = 3(-1) - 4 = -3 - 4 = -7$

$$f(2) = 2; f(0) = -4; \textit{ and } f(-1) = -7$$

Function notation tells you much more than the value of the independent variable. It also indicates a point on the graph. For example, in the above example, $f(-1) = -7$. This means the ordered pair (-1, -7) is a solution to $f(x) = 3x - 4$ and appears on the graphed line. You can use this information to write an equation for a function.

Example 8: *Write an equation for a line with* $m = 3.5$ *and* $f(-2) = 1$.

Solution: You know the slope and you know a point on the graph (-2, 1). Using the methods presented in this lesson, write the equation for the line.

Begin with slope-intercept form

	$y = mx + b$
Substitute the value for the slope.	$y = 3.5x + b$
Use the ordered pair to solve for b.	$1 = 3.5(-2) + b$
	$b = 8$
Rewrite the equation.	$y = 3.5x + 8$
or	$f(x) = 3.5x + 8$

Solve Real-World Problems Using Linear Models

Lets apply the methods we just learned to a few application problems that can be modeled using a linear relationship.

Example 9: *Nadia has \$200 in her savings account. She gets a job that pays \$7.50 per hour and she deposits all her earnings in her savings account. Write the equation describing this problem in slopeintercept form. How many hours would Nadia need to work to have \$500 in her account?*

Solution: Begin by defining the variables:

$y =$ amount of money in Nadias savings account

$x =$ number of hours

The problem gives the $y-$intercept and the slope of the equation.

We are told that Nadia has \$200 in her savings account, so $b = 200$.

We are told that Nadia has a job that pays \$7.50 per hour, so $m = 7.50$.

By substituting these values in slopeintercept form $y = mx + b$, we obtain $y = 7.5x + 200$.

To answer the question, substitute \$500 for the value of y and solve.

$$500 = 7.5x + 200 \Rightarrow 7.5x = 300 \Rightarrow x = 40$$

Nadia must work 40 hours if she is to have \$500 in her account.

Example 10: *A stalk of bamboo of the family Phyllostachys nigra grows at steady rate of 12 inches per day and achieves its full height of 720 inches in 60 days. Write the equation describing this problem in slopeintercept form. How tall is the bamboo 12 days after it started growing?*

Solution: Define the variables.

$y =$ the height of the bamboo plant in inches

$x =$ number of days

The problem gives the slope of the equation and a point on the line.

The bamboo grows at a rate of 12 inches per day, so $m = 12$.

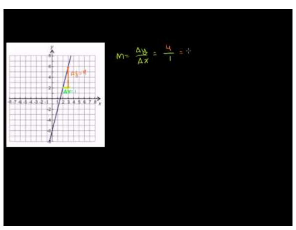
We are told that the plant grows to 720 inches in 60 days, so we have the point (60, 720).

Start with the slope-intercept form of the line.	$y = mx + b$
Substitute 12 for the slope.	$y = 12x + b$
Substitute the point $(60, 720)$.	$720 = 12(60) + b \Rightarrow b = 0$
Substitute the value of b back into the equation.	$y = 12x$

To answer the question, substitute the value $x = 12$ to obtain $y = 12(12) = 144$ inches.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. CK-12 Basic Algebra:Linear Equations inSlope InterceptForm (14:58)



MEDIA

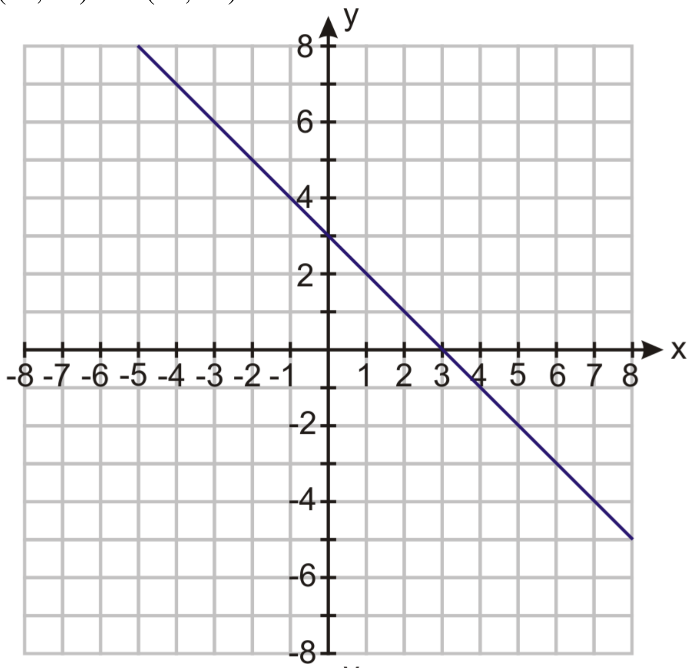
Click image to the left for more content.

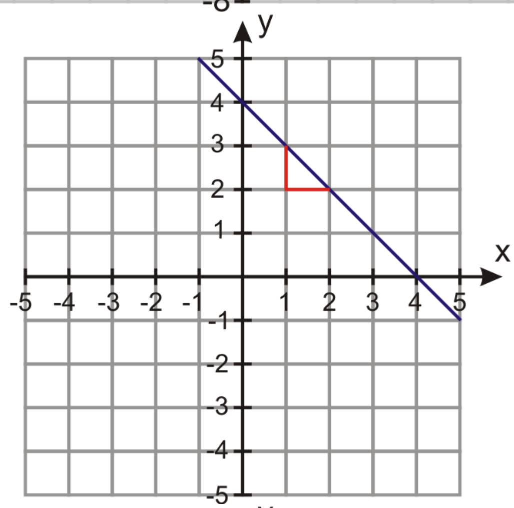
1. What is the formula for slope-intercept form? What do the variables m and b represent?
2. What are the five steps needed to determine the equation of a line given the slope and a point on the graph (not the $y-$intercept)?
3. What is the first step of finding the equation of a line given two points?

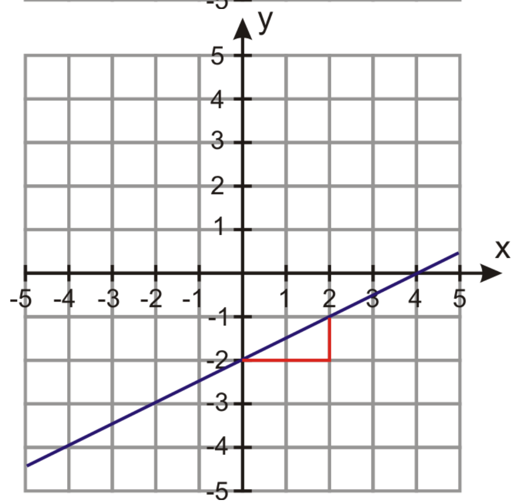
Find the equation of the line in slopeintercept form.

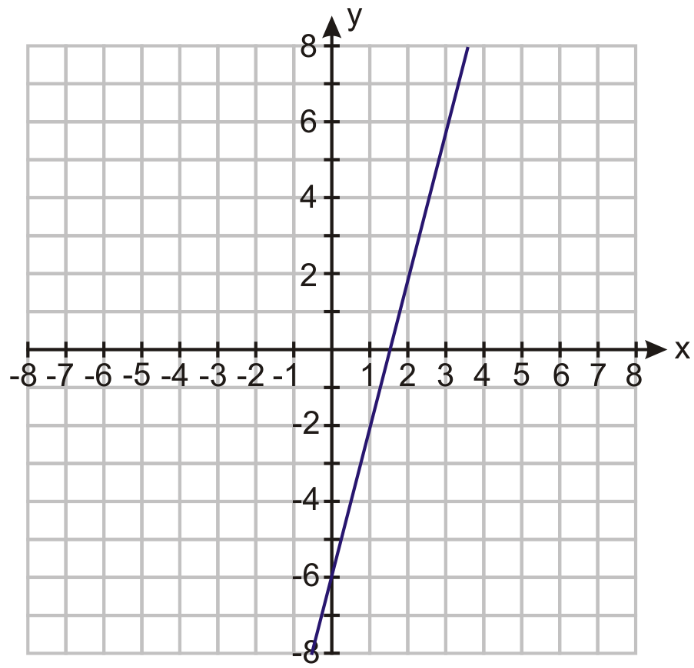
4. The line has slope of 7 and $y-$intercept of -2.
5. The line has slope of -5 and $y-$intercept of 6.
6. The line has slope $= -2$ and $ay-$intercept = 7.
7. The line has slope $= \frac{2}{3}$ and $ay-$intercept = $\frac{4}{5}$.
8. The line has slope of $-\frac{1}{4}$ and contains point (4, -1).
9. The line has slope of $\frac{2}{3}$ and contains point $\left(\frac{1}{2}, 1\right)$.
10. The line has slope of -1 and contains point $\left(\frac{4}{5}, 0\right)$.
11. The line contains points (2, 6) and (5, 0).
12. The line contains points (5, -2) and (8, 4).

13. The line contains points (3, 5) and (-3, 0).
14. The slope of the line is $-\frac{2}{3}$ and the line contains point (2, -2).
15. The slope of the line is 3 and the line contains point (3, -5).
16. The line contains points (10, 15) and (12, 20)

17. 

18. 

19. 

20. 

Find the equation of the linear function in slopeintercept form.

21. $m = 5, f(0) = -3$
22. $m = -2$ and $f(0) = 5$
23. $m = -7, f(2) = -1$
24. $m = \frac{1}{3}, f(-1) = \frac{2}{3}$
25. $m = 4.2, f(-3) = 7.1$
26. $f\left(\frac{1}{4}\right) = \frac{3}{4}, f(0) = \frac{5}{4}$
27. $f(1.5) = -3, f(-1) = 2$
28. $f(-1) = 1$ and $f(1) = -1$
29. To buy a car, Andrew puts a down payment of \$1500 and pays \$350 per month in installments. Write an equation describing this problem in slope-intercept form. How much money has Andrew paid at the end of one year?
30. Anne transplants a rose seedling in her garden. She wants to track the growth of the rose so she measures its height every week. On the third week, she finds that the rose is 10 inches tall and on the eleventh week she finds that the rose is 14 inches tall. Assuming the rose grows linearly with time, write an equation describing this problem in slope-intercept form. What was the height of the rose when Anne planted it?
31. Ravi hangs from a giant exercise spring whose length is 5 m. When his child Nimi hangs from the spring its length is 2 m. Ravi weighs 160 lbs. and Nimi weighs 40 lbs. Write the equation for this problem in slope-intercept form. What should we expect the length of the spring to be when his wife Amardeep, who weighs 140 lbs., hangs from it?
32. Petra is testing a bungee cord. She ties one end of the bungee cord to the top of a bridge and to the other end she ties different weights and measures how far the bungee stretches. She finds that for a weight of 100 lbs., the bungee stretches to 265 feet and for a weight of 120 lbs., the bungee stretches to 275 feet. Physics tells us that in a certain range of values, including the ones given here, the amount of stretch is a linear function of the weight. Write the equation describing this problem in slopeintercept form. What should we expect the stretched length of the cord to be for a weight of 150 lbs?

11.2 Writing Linear Equations in Standard Form

As the past few lessons of this chapter have shown, there are several ways to write a linear equation. This lesson introduces another method: **standard form**. You have already seen examples of standard form equations in a previous lesson. For example, here are some equations written in standard form.

$$\begin{aligned} 0.75(h)+1.25(b)&=30\\ 7x-3y&=21\\ 2x+3y&=-6 \end{aligned}$$

The **standard form** of a linear equation has the form $Ax+By=C$, *where* A,B, *and* C *are integers and* A *and* B *are* ***not*** *both zero*.

Example 1: Rewrite $y-5=3(x-2)$ in standard form.

Solution: Use the Distributive Property to simplify the right side of the equation

$$y-5=3x-6$$

Rewrite this equation so the variables x and y are on the same side of the equation.

$$\begin{aligned} y-5+6&=3x-6+6\\ y-y+1&=3x-y\\ 1&=3x-y, && \text{where A=3, B=-1, and C=1.} \end{aligned}$$

Example 2: Rewrite $5x-7=y$ in standard form.

Solution: Rewrite this equation so the variables x and y are on the same side of the equation.

$$\begin{aligned} 5x-7+7&=y+7\\ 5x-y&=y-y+7\\ 5x-y&=7, && \text{where A=5, B=-1, and C=7.} \end{aligned}$$

Finding Slope and Intercept of a Standard Form Equation

Slope-intercept form of a linear equation contains the slope of the equation explicitly, but the standard form does not. Since the slope is such an important feature of a line, it is useful to figure out how you would find the slope if you were given the equation of the line in standard form.

Begin with standard form: $Ax+By=C$.

If you rewrite this equation in slope-intercept form, it becomes

$$\begin{aligned}Ax - Ax + By &= C - Ax\\ \frac{By}{B} &= \frac{-Ax + C}{B}\\ y &= \frac{-A}{B}x + \frac{C}{B}\end{aligned}$$

When you compare this form to slope-intercept form, $y = mx + b$, you can see that the slope of a standard form equation is $\frac{-A}{B}$ and the $y-$intercept is $\frac{C}{B}$.

The **standard form** of a linear equation $Ax + By = C$ has the following:

$slope = \frac{-A}{B}$ and $y - intercept = \frac{C}{B}$.

Example 3: Find the slope and $y-$intercept of $2x - 3y = -8$.

Solution: Using the definition of standard form, $A = 2, B = -3, \text{ and } C = -8$.

$$\begin{aligned}slope = \frac{-A}{B} &= \frac{-2}{-3} \rightarrow \frac{2}{3}\\ y - intercept = \frac{C}{B} &= \frac{-8}{-3} \rightarrow \frac{8}{3}\end{aligned}$$

The slope is $\frac{2}{3}$ and the $y-$intercept is $\frac{8}{3}$.

Applying Standard Form to Real-World Situations

Example 4: *Nimitha buys fruit at her local farmers market. This Saturday, oranges cost \$2 per pound and cherries cost \$3 per pound. She has \$12 to spend on fruit. Write an equation in standard form that describes this situation. If she buys 4 pounds of oranges, how many pounds of cherries can she buy?*

Solution: Define the variables: $x =$ pounds of oranges and $y =$ pounds of cherries

The equation that describes this situation is: $2x + 3y = 12$

If she buys 4 pounds of oranges, we substitute $x = 4$ in the equation and solve for y.

$2(4)+3y=12 \Rightarrow 3y=12-8 \Rightarrow 3y=4 \Rightarrow y=\frac{4}{3}$. Nimitha can buy $1\frac{1}{3}$ pounds of cherries.

Example 5: *Jethro skateboards part of the way to school and walks for the rest of the way. He can skateboard at 7 miles per hour and he can walk at 3 miles per hour. The distance to school is 6 miles. Write an equation in standard form that describes this situation. If Jethro skateboards for* $\frac{1}{2}$ *an hour, how long does he need to walk to get to school?*

Solution: Define the variables: $x =$ hours Jethro skateboards and $y =$ hours Jethro walks

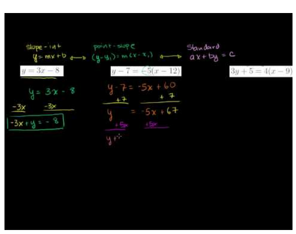
The equation that describes this situation is $7x+3y=6$.

If Jethro skateboards $\frac{1}{2}$ hour, we substitute $x=0.5$ in the equation and solve for y.

$7(0.5)+3y=6 \Rightarrow 3y=6-3.5 \Rightarrow 3y=2.5 \Rightarrow y=\frac{5}{6}$. Jethro must walk $\frac{5}{6}$ of an hour.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. CK-12 Basic Algebra:Linear Equations inStandard Form (10:08)



MEDIA

Click image to the left for more content.

1. What is the standard form of a linear equation? What do A, B, and C represent?
2. What is the meaning of clear the fractions? How would you go about doing so?

3. Consider the equation $Ax+By=C$. What are the slope and $y-$intercept of this equation?

Rewrite the following equations in standard form.

4. $y=3x-8$
5. $y=-x-6$
6. $y=\frac{5}{3}x-4$
7. $0.30x+0.70y=15$
8. $5=\frac{1}{6}x-y$
9. $y-7=-5(x-12)$
10. $2y=6x+9$
11. $y=\frac{9}{4}x+\frac{1}{4}$
12. $y+\frac{3}{5}=\frac{2}{3}(x-2)$
13. $3y+5=4(x-9)$

Find the slope and $y-$intercept of the following lines.

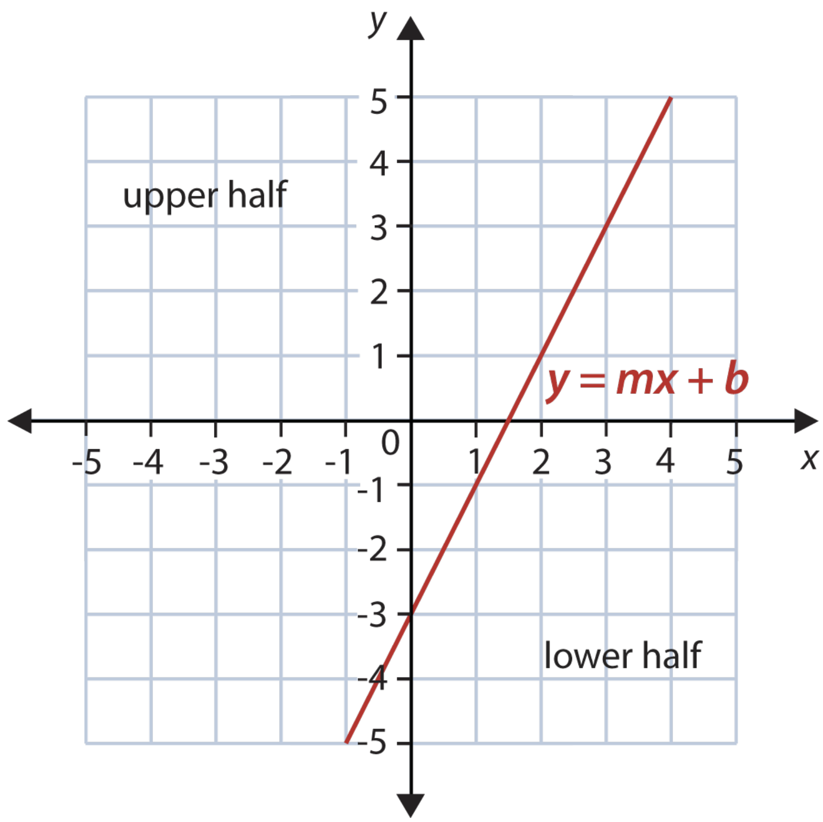
14. $5x-2y=15$
15. $3x+6y=25$
16. $x-8y=12$
17. $3x-7y=20$
18. $9x-9y=4$
19. $6x+y=3$
20. $x-y=9$
21. $8x+3y=15$
22. $4x+9y=1$

Write each equation in standard form.

23. The farmers market also sells tomatoes and corn. Tomatoes are selling for $1.29 per pound and corn is selling for $3.25 per pound. If you buy 6 pounds of tomatoes, how many pounds of corn can you buy if your total spending cash is $11.61?
24. The local church is hosting a Friday night fish fry for Lent. They sell a fried fish dinner for $7.50 and a baked fish dinner for $8.25. The church sold 130 fried fish dinners and took in $2,336.25. How many baked fish dinners were sold?
25. Andrew has two part time jobs. One pays $6 per hour and the other pays $10 per hour. He wants to make $366 per week. Write an equation in standard form that describes this situation. If he is only allowed to work 15 hours per week at the $10 per hour job, how many hours does he need to work per week at his $6 per hour job in order to achieve his goal?
26. Anne invests money in two accounts. One account returns 5% annual interest and the other returns 7% annual interest. In order not to incur a tax penalty, she can make no more than $400 in interest per year. Write an equation in standard form that describes this problem. If she invests $5000 in the 5% interest account, how much money does she need to invest in the other account?

11.3 Linear Inequalities in Two Variables

When a linear equation is graphed in a coordinate plane, the line splits the plane into two pieces. Each piece is called a **half-plane**. The diagram below shows how the half-planes are formed when graphing a linear equation.

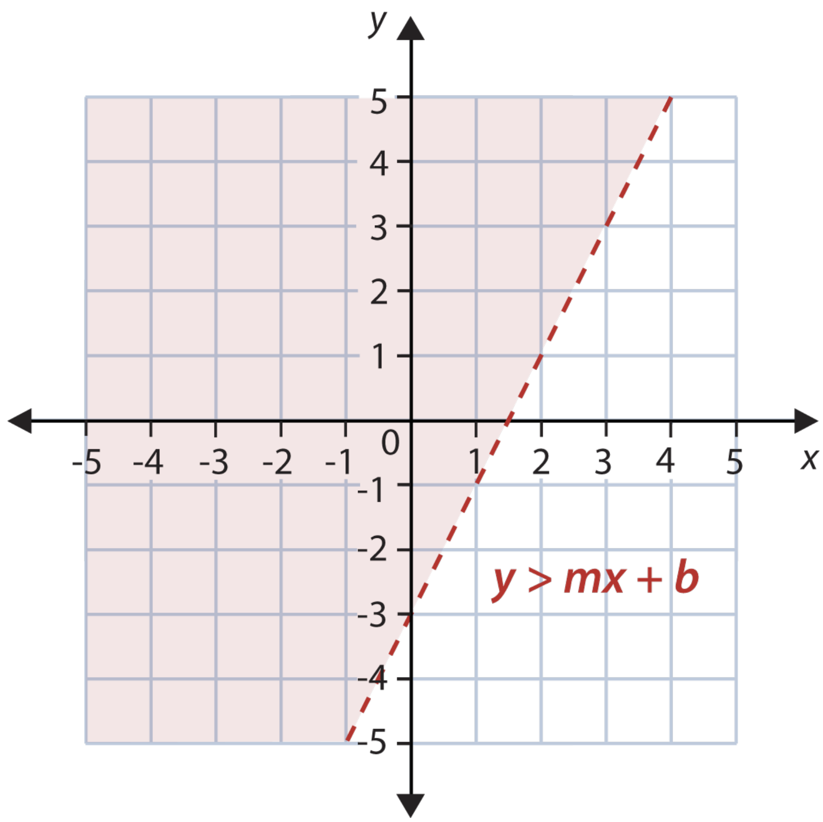


A linear inequality in two variables can also be graphed. Instead of only graphing the **boundary line** $(y = mx + b)$, you must also include all the other ordered pairs that could be solutions to the inequality. This is called the **solution set** and is shown by shading, or coloring the half-plane that includes the appropriate solutions.

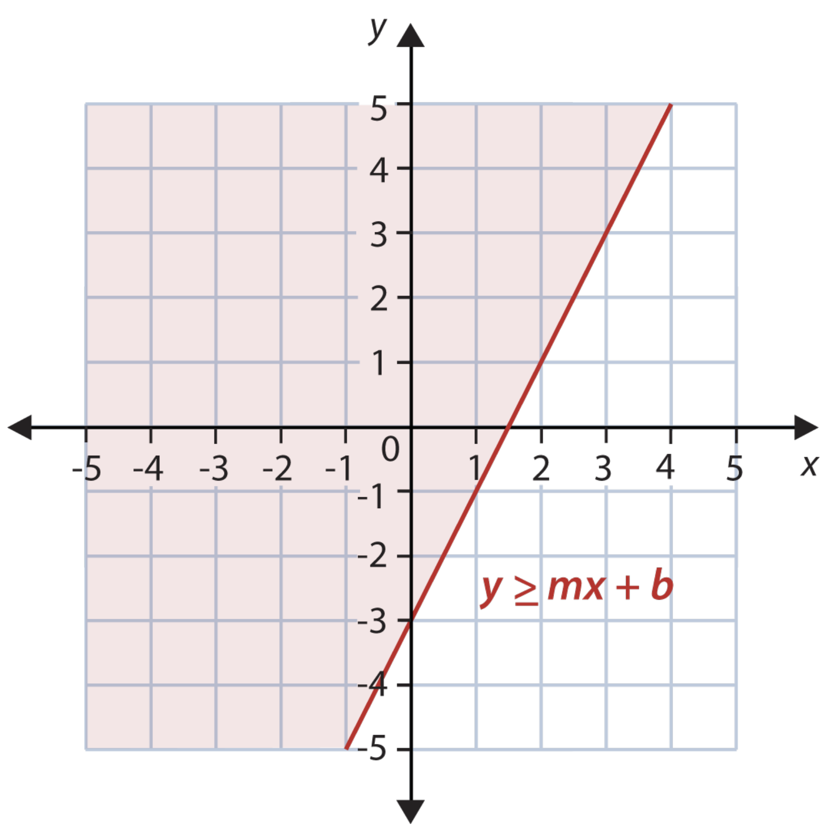
When graphing inequalities in two variables, you must be remember when the value is included $\leq or \geq$ or not included $< or >$. To represent these inequalities on a coordinate plane, instead of shaded or unshaded circles, we use solid and dashed lines. We can tell which half of the plane the solution is by looking at the inequality sign.

- $>$ The solution is the half plane above the line.
- $\geq$ The solution is the half plane above the line and also all the points on the line.
- $<$ The solution is the half plane below the line.
- $\leq$ The solution is the half plane below the line and also all the points on the line.

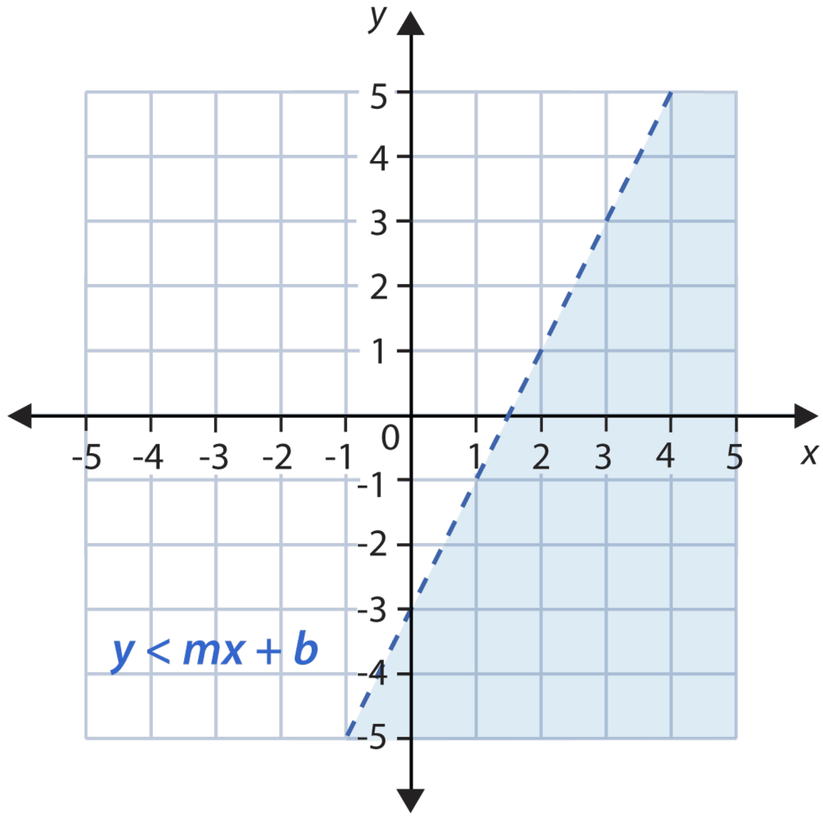
The solution of $y > mx + b$ is the half plane above the line. The dashed line shows that the points on the line are not part of the solution.



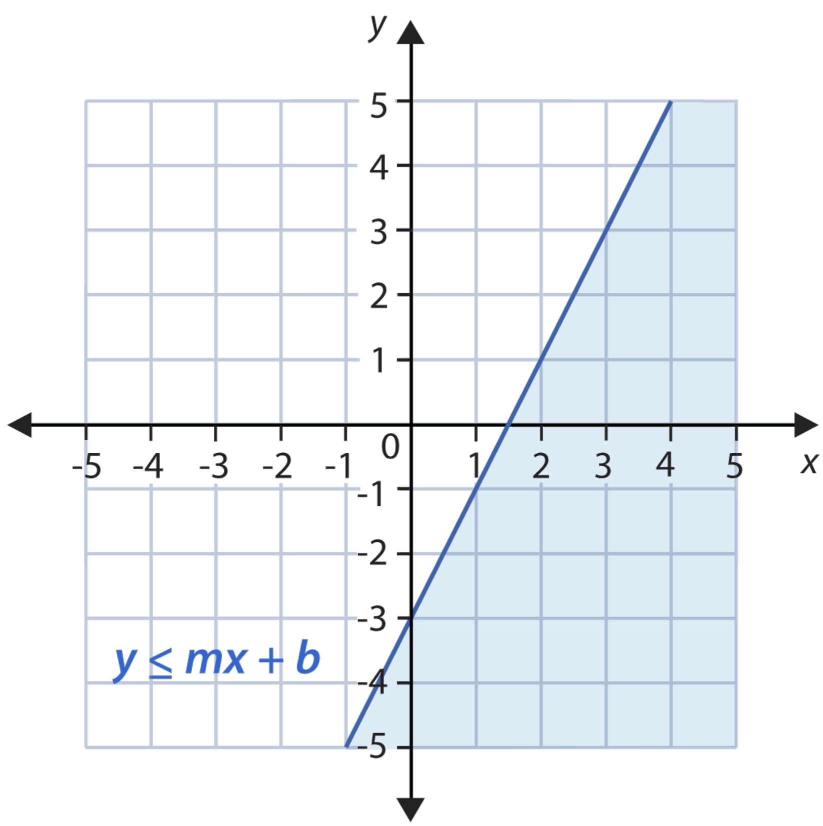
The solution of $y \geq mx+b$ is the half plane above the line and all the points on the line.



The solution of $y < mx+b$ is the half plane below the line.



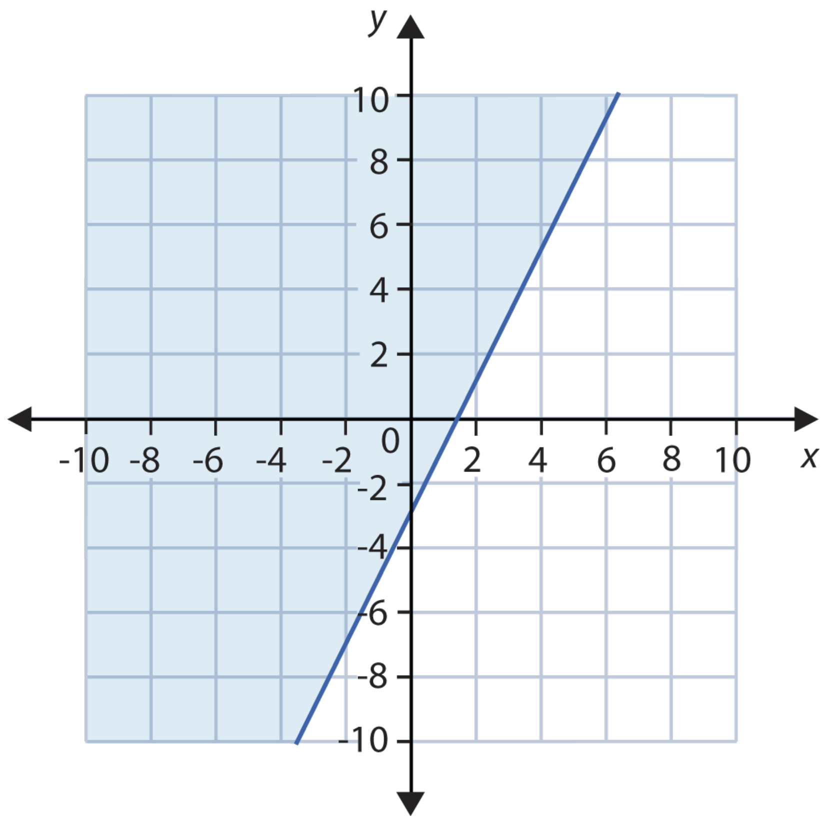
The solution of $y \leq mx + b$ is the half plane below the line and all the points on the line.



Example 1: *Graph the inequality* $y \geq 2x - 3$.

Solution: This inequality is in a slope-intercept form. Begin by graphing the line. Then determine the half-plane to color.

- The inequality is $\geq$, so the line is solid.
- The inequality states to shade the half-plane above the boundary line.



In general, the process used to graph a linear inequality in two variables is:

Step 1: Graph the equation using the most appropriate method.

- Slope-intercept form uses the $y-$intercept and slope to find the line
- Standard form uses the intercepts to graph the line
- Point-slope uses a point and the slope to graph the line

Step 2: If the equal sign is not included draw a dashed line. Draw a solid line if the equal sign is included.

Step 3: Shade the half plane above the line if the inequality is greater than. Shade the half plane under the line if the inequality is less than.

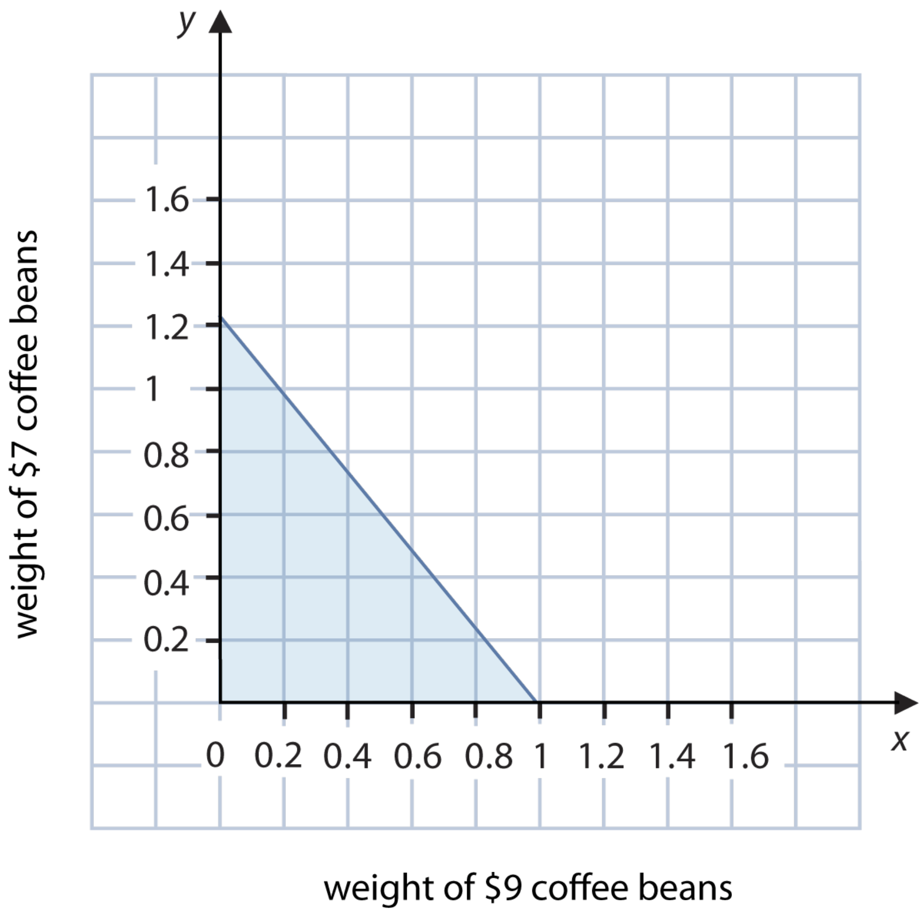
Example: *A pound of coffee blend is made by mixing two types of coffee beans. One type costs $9.00 per pound and another type costs $7.00 per pound. Find all the possible mixtures of weights of the two different coffee beans for which the blend costs $8.50 per pound or less.*

Solution: Begin by determining the appropriate letters to represent the varying quantities.

Let $x =$ weight of $9.00 per pound coffee beans in pounds and let $y =$ weight of $7.00 per pound coffee beans in pounds

Translate the information into an inequality. $9x + 7y \leq 8.50$.

Because the inequality is in standard form, it will be easier to graph using its intercepts.



When $x = 0, y = 1.21$. When $y = 0, x = 0.944$.

Graph the inequality. The line will be solid. We shade below the line.

We only graphed the first quadrant of the coordinate plane because neither bag should have a negative weight.

The blue-shaded region tells you all the possibilities of the two bean mixtures that will give a total less than or equal to $8.50.

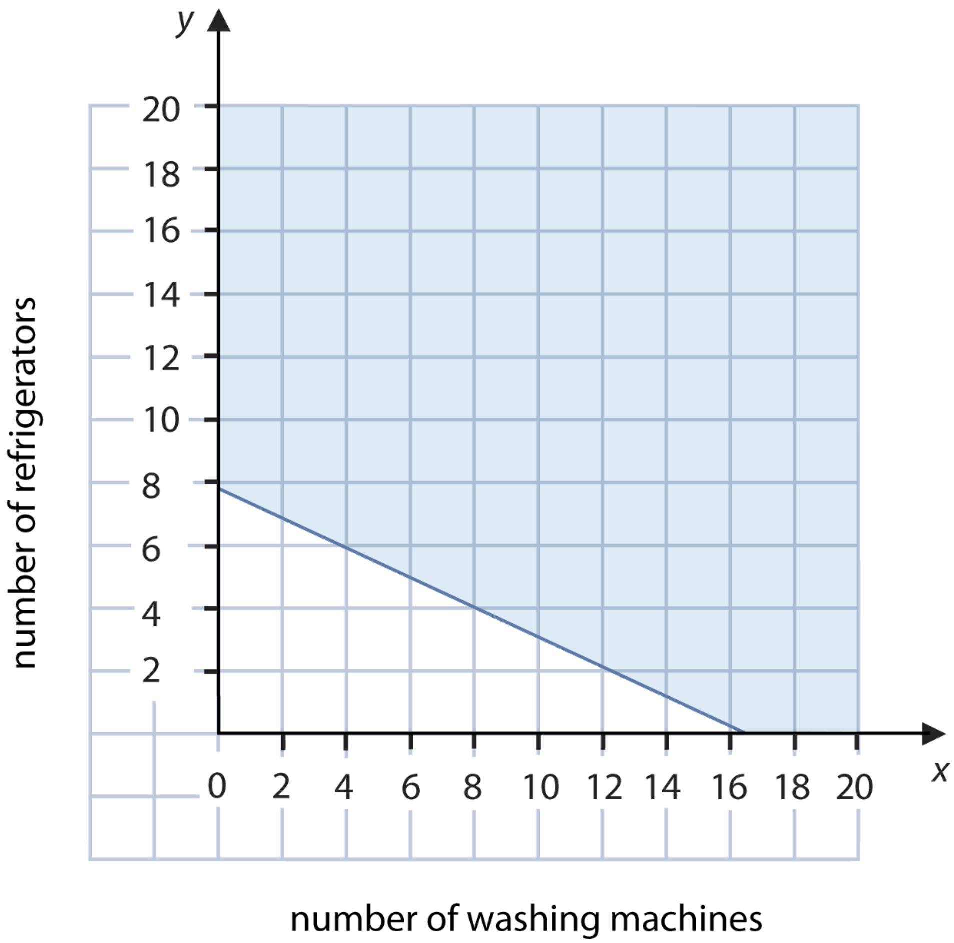
Example 2: *Julian has a job as an appliance salesman. He earns a commission of $60 for each washing machine he sells and $130 for each refrigerator he sells. How many washing machines and refrigerators must Julian sell in order to make $1,000 or more in commission?*

Solution: Determine the appropriate variables for the unknown quantities. Let $x =$ number of washing machines Julian sells and let $y =$ number of refrigerators Julian sells

Now translate the situation into an inequality. $60x + 130y \geq 1000$.

Graph the standard form inequality using its intercepts. When $x = 0, y = 16.667$. When $y = 0, x = 7.692$. The line will be solid.

We want the ordered pairs that are solutions to Julian making more than $10,000, so we shade the half-plane above the boundary line.

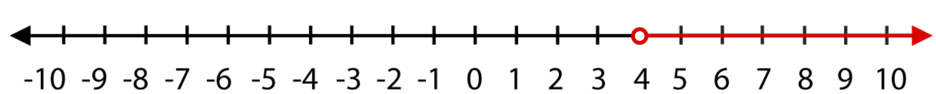


Graphing Horizontal and Vertical Linear Inequalities

Linear inequalities in one variable can also be graphed in the coordinate plane. They take the form of horizontal and vertical lines, however the process is identical to graphing **oblique**, or slanted, lines.

Example: *Graph the inequality* $x > 4$ *on* 1) *a number line and* 2) *the coordinate plane.*

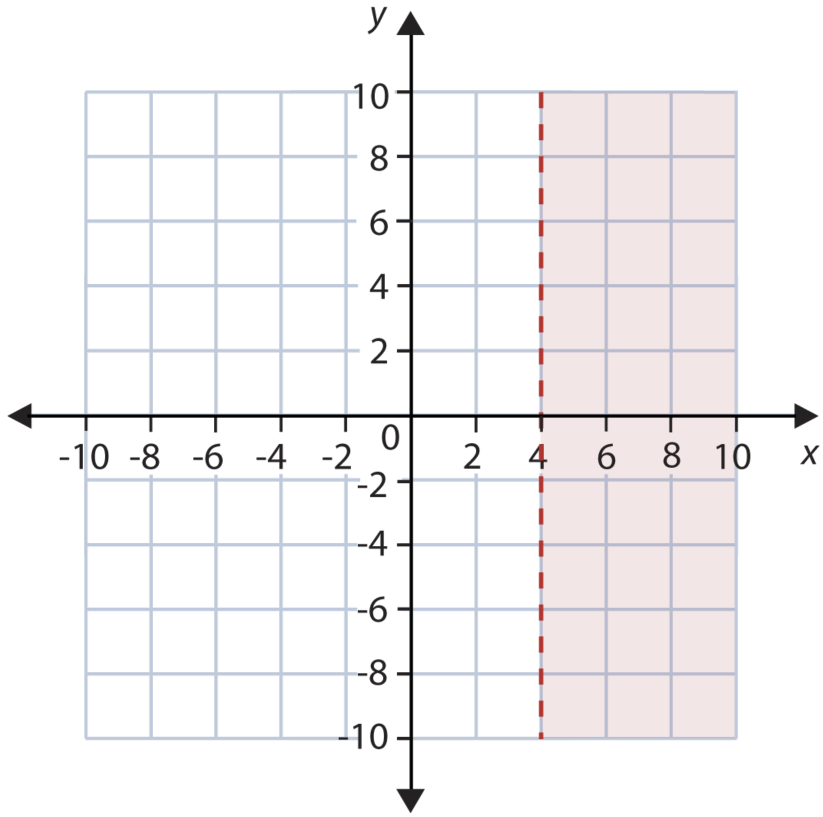
Solution: Remember what the solution to $x > 4$ looks like on a number line.



The solution to this inequality is the set of all real numbers x that are bigger than four but not including four.

On a coordinate plane, the line $x = 4$ is a vertical line four units to the right of the origin. The inequality **does not equal** four, so the vertical line is dashed this shows the reader the ordered pairs on the vertical line $x = 4$ are not solutions to the inequality.

The inequality is looking for all $x-$coordinates larger than four. We then color the half-plane to the right, symbolizing $x > 4$.



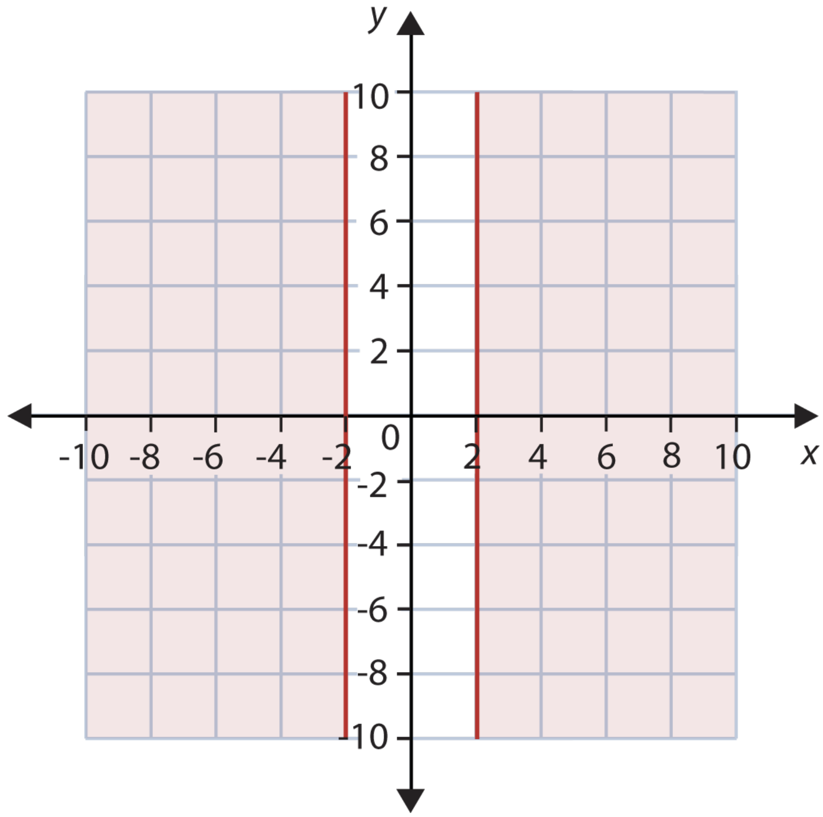
Graphing absolute value inequalities can also be done in the coordinate plane. To graph the inequality $|x| \geq 2$, we remember lesson six of this chapter and rewrite the absolute value inequality.

$x \leq -2$ or $x \geq 2$

Then graph each inequality on a coordinate plane.

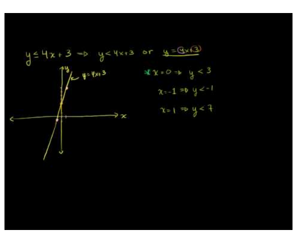
In other words, the solution is all the coordinate points for which the value of x is smaller than or equal to -2 and greater than or equal to 2. The solution is represented by the plane to the left of the vertical line $x = -2$ and the plane to the right of line $x = 2$.

Both vertical lines are solid because points on the line are included in the solution.



Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. CK-12 Basic Algebra:Graphing Inequalities (8:03)



MEDIA

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1. Define *half-plane.*
2. In which cases would the boundary line be represented by a dashed line?
3. In which cases would the boundary line be represented by a solid line?
4. What is a method to help you determine which half-plane to color?

Graph each inequality in a coordinate plane.

5. $x < 20$
6. $y \geq -5$
7. $y \leq 6$
8. $|x| > 10$
9. $|y| \leq 7$
10. $y \leq 4x + 3$
11. $y > -\frac{x}{2} - 6$
12. $y \leq -\frac{1}{2}x + 5$
13. $3x - 4y \geq 12$
14. $x + 7y < 5$
15. $y < -4x + 4$
16. $y > \frac{7}{2}x + 3$
17. $6x + 5y > 1$
18. $6x - 5y \leq 15$
19. $2x - y < 5$
20. $y + 5 \leq -4x$
21. $x - \frac{1}{2}y \geq 5$
22. $y \leq -\frac{x}{3} + 5$
23. $5x - 2y > 4$
24. $30x + 5y < 100$
25. $y \geq -x$
26. $6x - y < 4$
27. Lili can make yarn ankle bracelets and wrist bracelets. She has 600
28. yards of yarn available. It takes 6 yards to make one wrist bracelet and 8 yards to make one anklet. Find all the possible combinations of anklets and bracelets she can make without going over her available yarn.
29. An ounce of gold costs \$670 and an ounce of silver costs \$13. Find all possible weights of silver and gold that makes an alloy that costs less than \$600 per ounce.
30. A phone company charges 50 cents per minute during the daytime and 10 cents per minute at night. How many daytime minutes and night time minutes would you have to use to pay more than \$20.00 over a 24-hour period?
31. Jessie has \$30 to spend on food for a class barbeque. Hot dogs cost \$0.75 each (including the bun) and burgers cost \$1.25 (including bun and salad). Plot a graph that shows all the combinations of hot dogs and burgers he could buy for the barbecue, spending less than \$30.00.
32. At the local grocery store strawberries cost \$3.00 per pound and bananas cost \$1.00 per pound. If I have \$10 to spend between strawberries and bananas, draw a graph to show what combinations of each I can buy and spend at most \$10.

CHAPTER 12 Systems of Linear Equations

Chapter Outline

12.1 Solving Linear Systems by Graphing

Introduction

A linear system of equations is a set of equations that must be solved together to find the one solution that fits them both.

In this lesson, well discover methods to determine if an ordered pair is a solution to a system of two equations. Then well learn to solve the two equations graphically and confirm that the solution is the point where the two lines intersect. Finally, well look at real-world problems that can be solved using the methods described in this chapter.

Determine Whether an Ordered Pair is a Solution to a System of Equations

Consider this system of equations:

$$y = x + 2$$
$$y = -2x + 1$$

Since the two lines are in a system, we deal with them together by graphing them on the same coordinate axes. We can use any method to graph them; lets do it by making a table of values for each line.

Line 1: $y = x + 2$

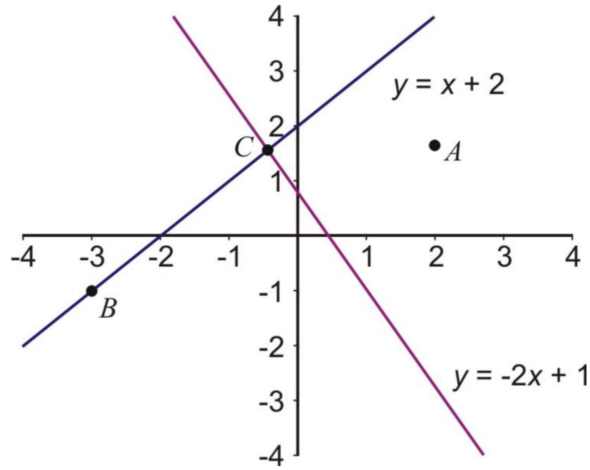
TABLE 12.1:

x	y
0	2
1	3

Line 2: $y = -2x + 1$

TABLE 12.2:

x	y
0	1
1	-1



We already know that any point that lies on a line is a solution to the equation for that line. That means that any point that lies on *both* lines in a system is a solution to both equations.

So in this system:

- Point A is not a solution to the system because it does not lie on either of the lines.
- Point B is not a solution to the system because it lies only on the blue line but not on the red line.
- Point C is a solution to the system because it lies on both lines at the same time.

In fact, point C is the only solution to the system, because it is the only point that lies on both lines. For a system of equations, the geometrical solution is the intersection of the two lines in the system. The algebraic solution is the ordered pair that solves both equationsin other words, the coordinates of that intersection point.

You can confirm the solution by plugging it into the system of equations, and checking that the solution works in each equation.

Example 1

Determine which of the points (1, 3), (0, 2), or (2, 7) is a solution to the following system of equations:

$$\begin{aligned} y &= 4x - 1 \\ y &= 2x + 3 \end{aligned}$$

Solution

To check if a coordinate point is a solution to the system of equations, we plug each of the x and y values into the equations to see if they work.

Point (1, 3):

$$\begin{aligned} y &= 4x - 1 \\ 3 &\stackrel{?}{=} 4(1) - 1 \\ 3 &= 3 \text{ solution checks} \end{aligned}$$

$$\begin{aligned} y &= 2x + 3 \\ 3 &\stackrel{?}{=} 2(1) + 3 \\ 3 &\neq 5 \text{ solution does not check} \end{aligned}$$

Point (1, 3) is on the line $y = 4x - 1$, but it is not on the line $y = 2x + 3$, so it is not a solution to the system.

Point (0, 2):

$$\begin{aligned} y &= 4x - 1 \\ 2 &\stackrel{?}{=} 4(0) - 1 \\ 2 &\neq -1 \text{ solution does not check} \end{aligned}$$

Point (0, 2) is not on the line $y = 4x - 1$, so it is not a solution to the system. Note that it is not necessary to check the second equation because the point needs to be on both lines for it to be a solution to the system.

Point (2, 7):

$$\begin{aligned} y &= 4x - 1 \\ 7 &\stackrel{?}{=} 4(2) - 1 \\ 7 &= 7 \text{ solution checks} \end{aligned}$$

$$\begin{aligned} y &= 2x + 3 \\ 7 &\stackrel{?}{=} 2(2) + 3 \\ 7 &= 7 \text{ solution checks} \end{aligned}$$

Point (2, 7) is a solution to the system since it lies on both lines.

The solution to the system is the point (2, 7).

Determine the Solution to a Linear System by Graphing

The solution to a linear system of equations is the point, (if there is one) that lies on both lines. In other words, the solution is the point where the two lines intersect.

We can solve a system of equations by graphing the lines on the same coordinate plane and reading the intersection point from the graph.

This method most often offers only approximate solutions, so its not sufficient when you need an exact answer. However, graphing the system of equations can be a good way to get a sense of whats really going on in the problem youre trying to solve, especially when its a real-world problem.

Example 2

Solve the following system of equations by graphing:

$$\begin{aligned} y &= 3x - 5 \\ y &= -2x + 5 \end{aligned}$$

Solution

Graph both lines on the same coordinate axis using any method you like.

In this case, lets make a table of values for each line.

Line 1: $y = 3x - 5$

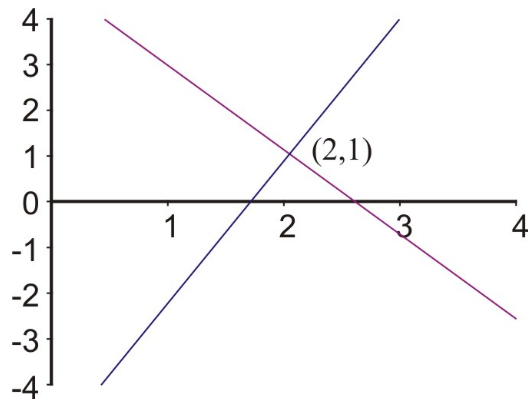
TABLE 12.3:

x	y
1	-2
2	1

Line 2: $y = -2x + 5$

TABLE 12.4:

x	y
1	3
2	1



The solution to the system is given by the intersection point of the two lines. The graph shows that the lines intersect at point (2, 1). So **the solution is** $x = 2, y = 1$**or (2, 1).**

Example 3

Solve the following system of equations by graphing:

$$\begin{aligned} 2x + 3y &= 6 \\ 4x - y &= -2 \end{aligned}$$

Solution

Since the equations are in standard form, this time well graph them by finding the $x-$ and $y-$intercepts of each of the lines.

Line 1: $2x + 3y = 6$

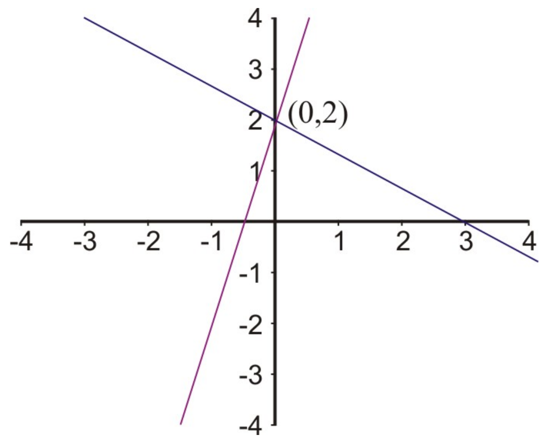
$x-$intercept: set $y = 0 \Rightarrow 2x = 6 \Rightarrow x = 3$ so the intercept is (3, 0)

$y-$intercept: set $x = 0 \Rightarrow 3y = 6 \Rightarrow y = 2$ so the intercept is (0, 2)

Line 2: $-4x + y = 2$

$x-$intercept: set $y = 0 \Rightarrow -4x = 2 \Rightarrow x = -\frac{1}{2}$ so the intercept is $\left(-\frac{1}{2}, 0\right)$

$y-$intercept: set $x = 0 \Rightarrow y = 2$ so the intercept is (0, 2)



The graph shows that the lines intersect at (0, 2). Therefore, **the solution to the system of equations is** $x = 0, y = 2$.

Solving a System of Equations Using a Graphing Calculator

As an alternative to graphing by hand, you can use a graphing calculator to find or check solutions to a system of equations.

Example 4

Solve the following system of equations using a graphing calculator.

$$\begin{aligned} x - 3y &= 4 \\ 2x + 5y &= 8 \end{aligned}$$

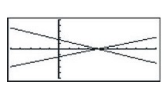
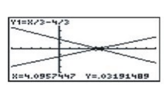
To input the equations into the calculator, you need to rewrite them in slope-intercept form (that is, $y = mx + b$ form).

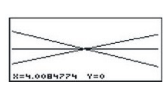
$$\begin{array}{lcl} x - 3y = 4 & & y = \frac{1}{3}x - \frac{4}{3} \\ & \Rightarrow & \\ 2x + 5y = 8 & & y = -\frac{2}{5}x + \frac{8}{5} \end{array}$$

Press the **[y=]** button on the graphing calculator and enter the two functions as:

$$\begin{aligned} Y_1 &= \frac{x}{3} - \frac{4}{3} \\ Y_2 &= \frac{-2x}{5} + \frac{8}{5} \end{aligned}$$

Now press **[GRAPH]**. Heres what the graph should look like on a TI-83 family graphing calculator with the window set to $-5 \le x \le 10$ and $-5 \le y \le 5$.



There are a few different ways to find the intersection point.

Option 1: Use **[TRACE]** and move the cursor with the arrows until it is on top of the intersection point. The values of the coordinate point will be shown on the bottom of the screen. The second screen above shows the values to be $X = 4.0957447$ and $Y = 0.03191489$.

Use the **[ZOOM]** function to zoom into the intersection point and find a more accurate result. The third screen above shows the system of equations after zooming in several times. A more accurate solution appears to be $X = 4$ and $Y = 0$.

Option 2 Look at the table of values by pressing **[2nd] [GRAPH]**. The first screen below shows a table of values for this system of equations. Scroll down until the $Y-$values for the two functions are the same. In this case this occurs at $X = 4$ and $Y = 0$.

(Use the **[TBLSET]** function to change the starting value for your table of values so that it is close to the intersection point and you dont have to scroll too long. You can also improve the accuracy of the solution by setting the value of Δ Table smaller.)

Option 3 Using the **[2nd] [TRACE]** function gives the second screen shown above.

Scroll down and select intersect.

The calculator will display the graph with the question **[FIRSTCURVE]**? Move the cursor along the first curve until it is close to the intersection and press **[ENTER]**.

The calculator now shows **[SECONDCURVE]**?

Move the cursor to the second line (if necessary) and press **[ENTER]**.

The calculator displays **[GUESS]**?

Press **[ENTER]** and the calculator displays the solution at the bottom of the screen (see the third screen above).

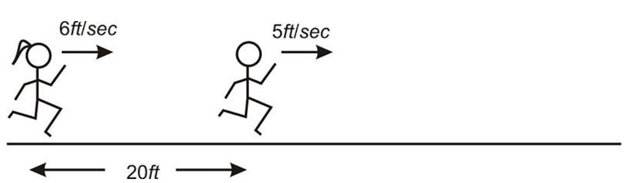
The point of intersection is $X = 4$ and $Y = 0$. Note that with this method, the calculator works out the intersection point for you, which is generally more accurate than your own visual estimate.

Solve Real-World Problems Using Graphs of Linear Systems

Consider the following problem:

Peter and Nadia like to race each other. Peter can run at a speed of 5 feet per second and Nadia can run at a speed of 6 feet per second. To be a good sport, Nadia likes to give Peter a head start of 20 feet. How long does Nadia take to catch up with Peter? At what distance from the start does Nadia catch up with Peter?

Lets start by drawing a sketch. Heres what the race looks like when Nadia starts running; well call this time $t = 0$.



Now lets define two variables in this problem:

$t =$ the time from when Nadia starts running

$d =$ the distance of the runners from the starting point.

Since there are two runners, we need to write equations for each of them. That will be the *system of equations* for this problem.

For each equation, we use the formula: distance $=$ speed $\times$ time

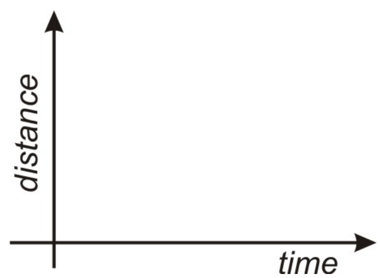
Nadias equation: $d = 6t$

Peters equation: $d = 5t + 20$

(Remember that Peter was already 20 feet from the starting point when Nadia started running.)

Lets graph these two equations on the same coordinate axes.

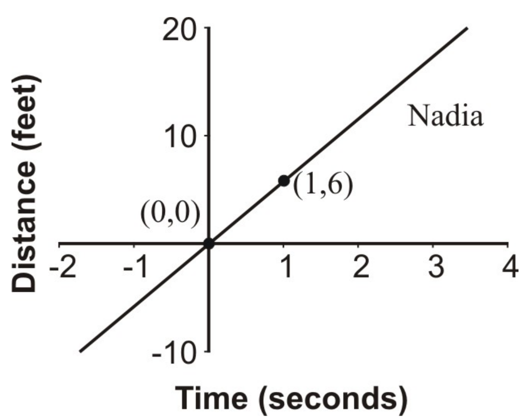
Time should be on the horizontal axis since it is the independent variable. Distance should be on the vertical axis since it is the dependent variable.



We can use any method for graphing the lines, but in this case well use the *slopeintercept* method since it makes more sense physically.

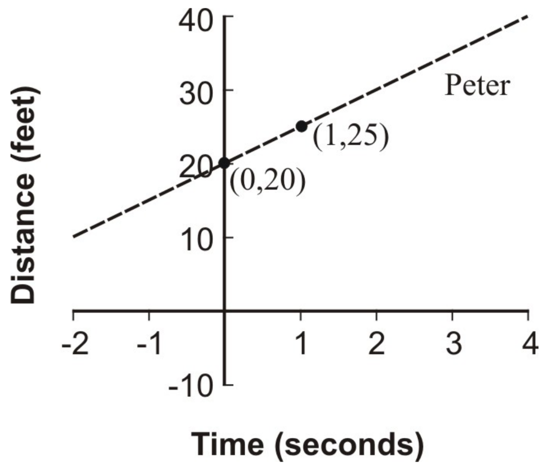
To graph the line that describes Nadias run, start by graphing the $y-$intercept: (0, 0). (If you dont see that this is the $y-$intercept, try plugging in the test-value of $x = 0$.)

The slope tells us that Nadia runs 6 feet every one second, so another point on the line is (1, 6). Connecting these points gives us Nadias line:

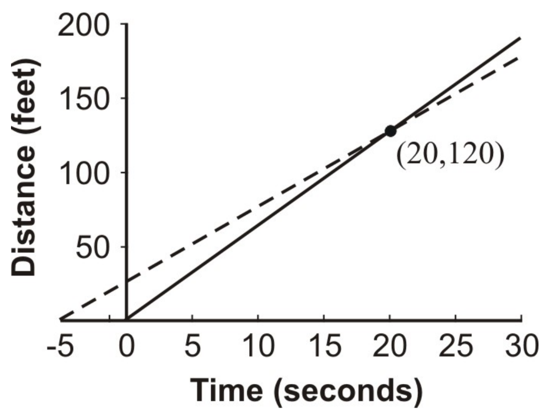


To graph the line that describes Peters run, again start with the $y-$intercept. In this case this is the point (0, 20).

The slope tells us that Peter runs 5 feet every one second, so another point on the line is (1, 25). Connecting these points gives us Peters line:



In order to find when and where Nadia and Peter meet, well graph both lines on the same graph and extend the lines until they cross. The crossing point is the solution to this problem.



The graph shows that Nadia and Peter meet **20 seconds after Nadia starts running, and 120 feet from the starting point.**

These examples are great at demonstrating that the solution to a system of linear equations means the point at which the lines intersect. This is, in fact, the greatest strength of the graphing method because it offers a very visual representation of system of equations and its solution. You can also see, though, that finding the solution from a graph requires very careful graphing of the lines, and is really only practical when youre sure that the solution gives integer values for x and y. Usually, this method can only offer approximate solutions to systems of equations, so we need to use other methods to get an exact solution.

Practice Set

Determine which ordered pair satisfies the system of linear equations.

1.

$$y = 3x - 2$$
$$y = -x$$

(a) $(1, 4)$
(b) $(2, 9)$
(c) $\left(\frac{1}{2}, \frac{-1}{2}\right)$

2.

$$y = 2x - 3$$
$$y = x + 5$$

(a) (8, 13)
(b) (-7, 6)
(c) (0, 4)

3.

$$2x + y = 8$$
$$5x + 2y = 10$$

(a) (-9, 1)
(b) (-6, 20)
(c) (14, 2)

4.

$$3x + 2y = 6$$
$$y = \frac{1}{2}x - 3$$

(a) $\left(3, \frac{-3}{2}\right)$
(b) (-4, 3)
(c) $\left(\frac{1}{2}, 4\right)$

5.

$$2x - y = 10$$
$$3x + y = -5$$

(a) (4, -2)
(b) (1, -8)
(c) (-2, 5)

Solve the following systems using the graphing method.

6.

$$y = x + 3$$
$$y = -x + 3$$

7.

$$y = 3x - 6$$
$$y = -x + 6$$

8.

$$2x = 4$$
$$y = -3$$

9.

$$\begin{aligned} y &= -x+5 \\ -x+y &= 1 \end{aligned}$$

10.

$$\begin{aligned} x+2y &= 8 \\ 5x+2y &= 0 \end{aligned}$$

11.

$$\begin{aligned} 3x+2y &= 12 \\ 4x-y &= 5 \end{aligned}$$

12. $5x+2y=-4$
 $x-y=2$
13. $2x+4=3y$
 $x-2y+4=0$
14.

$$\begin{aligned} y &= \frac{1}{2}x-3 \\ 2x-5y &= 5 \end{aligned}$$

15.

$$\begin{aligned} y &= 4 \\ x &= 8-3y \end{aligned}$$

16. Try to solve the following system using the graphing method:
 $y=\frac{3}{5}x+5$
 $y=-2x-\frac{1}{2}$.
 (a) What does it look like the $x-$coordinate of the solution should be?
 (b) Does that coordinate really give the same $y-$value when you plug it into both equations?
 (c) Why is it difficult to find the real solution to this system?
17. Try to solve the following system using the graphing method:
 $y=4x+8$
 $y=5x+1$.
 Use a grid with $x-$values and $y-$values ranging from -10 to 10.
 (a) Do these lines appear to intersect?
 (b) Based on their equations, are they parallel?
 (c) What would we have to do to find their intersection point?
18. Try to solve the following system using the graphing method:
 $y=\frac{1}{2}x+4$
 $y=\frac{4}{9}x+\frac{9}{2}$.
 Use the same grid as before.
 (a) Can you tell exactly where the lines cross?

(b) What would we have to do to make it clearer?

Solve the following problems by using the graphing method.

19. Marys car has broken down and it will cost her $1200 to get it fixedor, for $4500, she can buy a new, more efficient car instead. Her present car uses about $2000 worth of gas per year, while gas for the new car would cost about $1500 per year. After how many years would the total cost of fixing the car equal the total cost of replacing it?
20. Juan is considering two cell phone plans. The first company charges $120 for the phone and $30 per month for the calling plan that Juan wants. The second company charges $40 for the same phone but charges $45 per month for the calling plan that Juan wants. After how many months would the total cost of the two plans be the same?
21. A tortoise and hare decide to race 30 feet. The hare, being much faster, decides to give the tortoise a 20 foot head start. The tortoise runs at 0.5 feet/sec and the hare runs at 5.5 feet per second. How long until the hare catches the tortoise?

12.2 Solving Linear Systems by Substitution

Introduction

In this lesson, well learn to solve a system of two equations using the method of substitution.

Solving Linear Systems Using Substitution of Variable Expressions

Lets look again at the problem about Peter and Nadia racing.

Peter and Nadia like to race each other. Peter can run at a speed of 5 feet per second and Nadia can run at a speed of 6 feet per second. To be a good sport, Nadia likes to give Peter a head start of 20 feet. How long does Nadia take to catch up with Peter? At what distance from the start does Nadia catch up with Peter?

In that example we came up with two equations:

Nadias equation: $d = 6t$

Peters equation: $d = 5t + 20$

Each equation produced its own line on a graph, and to solve the system we found the point at which the lines intersectedthe point where the values for d and t satisfied **both** relationships. When the values for d and t are equal, that means that Peter and Nadia are at the same place at the same time.

But theres a faster way than graphing to solve this system of equations. Since we want the value of d to be the same in both equations, we could just set the two right-hand sides of the equations equal to each other to solve for t. That is, if $d = 6t$ and $d = 5t + 20$, and the two ds are equal to each other, then by the transitive property we have $6t = 5t + 20$. We can solve this for t:

$$\begin{aligned}
6t &= 5t + 20 && subtract\ 5t\ from\ both\ sides: \\
t &= 20 && substitute\ this\ value\ for\ t\ into\ Nadia's\ equation: \\
d &= 6 \cdot 20 = 120
\end{aligned}$$

Even if the equations werent so obvious, we could use simple algebraic manipulation to find an expression for one variable in terms of the other. If we rearrange Peters equation to isolate t:

$$\begin{aligned}
d &= 5t + 20 && subtract\ 20\ from\ both\ sides: \\
d - 20 &= 5t && divide\ by\ 5: \\
\frac{d - 20}{5} &= t
\end{aligned}$$

We can now ***substitute*** this expression for t into Nadias equation $(d = 6t)$ to solve:

$$\begin{aligned}
d &= 6\left(\frac{d-20}{5}\right) && \textit{multiply both sides by } 5: \\
5d &= 6(d-20) && \textit{distribute the } 6: \\
5d &= 6d-120 && \textit{subtract } 6d \textit{ from both sides}: \\
-d &= -120 && \textit{divide by } -1: \\
d &= 120 && \textit{substitute value for } d \textit{ into our expression for } t: \\
t &= \frac{120-20}{5} = \frac{100}{5} = 20
\end{aligned}$$

So we find that Nadia and Peter meet 20 seconds after they start racing, at a distance of 120 feet away.

The method we just used is called the **Substitution Method.** In this lesson youll learn several techniques for isolating variables in a system of equations, and for using those expressions to solve systems of equations that describe situations like this one.

Example 1

Lets look at an example where the equations are written in **standard form.**

Solve the system

$$\begin{aligned}
2x+3y &= 6 \\
-4x+y &= 2
\end{aligned}$$

Again, we start by looking to isolate one variable in either equation. If you look at the second equation, you should see that the coefficient of y is 1. So the easiest way to start is to use this equation to solve for y.

Solve the second equation for y:

$$\begin{aligned}
-4x+y &= 2 && \textit{add } 4x \textit{ to both sides}: \\
y &= 2+4x
\end{aligned}$$

Substitute this expression into the first equation:

$$\begin{aligned}
2x+3(2+4x) &= 6 && \textit{distribute the } 3: \\
2x+6+12x &= 6 && \textit{collect like terms}: \\
14x+6 &= 6 && \textit{subtract } 6 \textit{ from both sides}: \\
14x &= 0 && \textit{and hence}: \\
x &= 0
\end{aligned}$$

Substitute back into our expression for y:

$$y = 2+4\cdot 0 = 2$$

As you can see, we end up with the same solution $(x=0, y=2)$ that we found when we graphed these functions back in a previous lesson. So long as you are careful with the algebra, the substitution method can be a very efficient way to solve systems.

Next, lets look at a more complicated example. Here, the values of x and y we end up with arent whole numbers, so they would be difficult to read off a graph!

Example 2

Solve the system

$$\begin{aligned}2x+3y&=3\\2x-3y&=-1\end{aligned}$$

Again, we start by looking to isolate one variable in either equation. In this case it doesnt matter which equation we useall the variables look about equally easy to solve for.

So lets solve the first equation for x:

$$\begin{aligned}2x+3y&=3 && \textit{subtract } 3y \textit{ from both sides}:\\2x&=3-3y && \textit{divide both sides by } 2:\\x&=\frac{1}{2}(3-3y)\end{aligned}$$

Substitute this expression into the second equation:

$$\begin{aligned}\cancel{2}\cdot\frac{1}{\cancel{2}}(3-3y)-3y&=-1 && \textit{cancel the fraction and re-write terms}:\\3-3y-3y&=-1 && \textit{collect like terms}:\\3-6y&=-1 && \textit{subtract } 3 \textit{ from both sides}:\\-6y&=-4 && \textit{divide by } -6:\\y&=\frac{2}{3}\end{aligned}$$

Substitute into the expression we got for x:

$$\begin{aligned}x&=\frac{1}{2}\left(3-\cancel{3}\left(\frac{2}{\cancel{3}}\right)\right)\\x&=\frac{1}{2}\end{aligned}$$

So our solution is $x=\frac{1}{2}, y=\frac{2}{3}$. You can see how the graphical solution $\left(\frac{1}{2},\frac{2}{3}\right)$ might have been difficult to read accurately off a graph!

Solving Real-World Problems Using Linear Systems

Simultaneous equations can help us solve many real-world problems. We may be considering a purchasefor example, trying to decide whether its cheaper to buy an item online where you pay shipping or at the store where you do not. Or you may wish to join a CD music club, but arent sure if you would really save any money by buying a new CD every month in that way. Or you might be considering two different phone contracts. Lets look at an example of that now.

Example 3

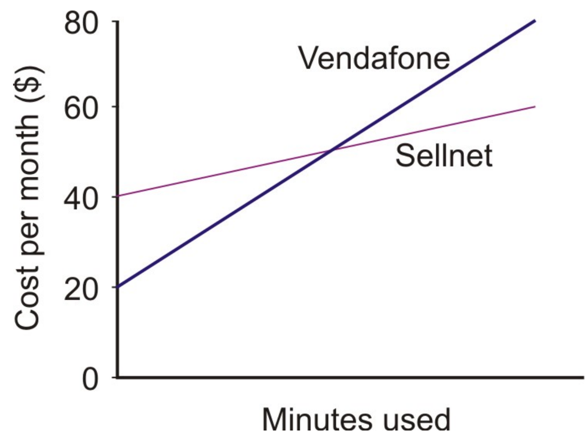
Anne is trying to choose between two phone plans. The first plan, with Vendafone, costs \$20 per month, with calls costing an additional 25 cents per minute. The second company, Sellnet, charges \$40 per month, but calls cost only 8 cents per minute. Which should she choose?

You should see that Annes choice will depend upon how many minutes of calls she expects to use each month. We start by writing two equations for the cost in dollars in terms of the minutes used. Since the *number of minutes* is the independent variable, it will be our x. Cost is *dependent* on minutes the *cost per month* is the dependent variable and will be assigned y.

For Vendafone: $y = 0.25x + 20$

For Sellnet: $y = 0.08x + 40$

By writing the equations in slope-intercept form $(y = mx + b)$, you can sketch a graph to visualize the situation:



The line for Vendafone has an intercept of 20 and a slope of 0.25. The Sellnet line has an intercept of 40 and a slope of 0.08 (which is roughly a third of the Vendafone lines slope). In order to help Anne decide which to choose, well find where the two lines cross, by solving the two equations as a system.

Since equation 1 gives us an expression for $y(0.25x + 20)$, we can substitute this expression directly into equation 2:

$$\begin{aligned}
0.25x + 20 &= 0.08x + 40 && \textit{subtract } 20 \textit{ from both sides}: \\
0.25x &= 0.08x + 20 && \textit{subtract } 0.08x \textit{ from both sides}: \\
0.17x &= 20 && \textit{divide both sides by } 0.17: \\
x &= 117.65 \textit{ minutes} && \textit{rounded to } 2 \textit{ decimal places}.
\end{aligned}$$

So if Anne uses 117.65 minutes a month (although she cant really do *exactly* that, because phone plans only count whole numbers of minutes), the phone plans will cost the same. Now we need to look at the graph to see which plan is better if she uses more minutes than that, and which plan is better if she uses fewer. You can see that the Vendafone plan costs more when she uses more minutes, and the Sellnet plan costs more with fewer minutes.

So, **if Anne will use 117 minutes or less every month she should choose *Vendafone.* If she plans on using 118 or more minutes she should choose *Sellnet.***

Mixture Problems

Systems of equations crop up frequently in problems that deal with mixtures of two thingschemicals in a solution, nuts and raisins, or even the change in your pocket! Lets look at some examples of these.

Example 4

Janine empties her purse and finds that it contains only nickels (worth 5 cents each) and dimes (worth 10 cents each). If she has a total of 7 coins and they have a combined value of 45 cents, how many of each coin does she have?

Since we have 2 types of coins, lets call the number of nickels x and the number of dimes y. We are given two key pieces of information to make our equations: the number of coins and their value.

$$\begin{aligned} &\text{of coins equation:} && x+y=7 && (number\ of\ nickels)+(number\ of\ dimes) \\ &\text{value equation:} && 5x+10y=55 && (since\ nickels\ are\ worth\ 5c\ and\ dimes\ 10c) \end{aligned}$$

We can quickly rearrange the first equation to isolate x:

$$\begin{aligned} &x=7-y && now\ substitute\ into\ equation\ 2: \\ &5(7-y)+10y=55 && distribute\ the\ 5: \\ &35-5y+10y=55 && collect\ like\ terms: \\ &35+5y=55 && subtract\ 35\ from\ both\ sides: \\ &5y=20 && divide\ by\ 5: \\ &\underline{y=4} && substitute\ back\ into\ equation\ 1: \\ &x+4=7 && subtract\ 4\ from\ both\ sides: \\ &\underline{x=3} \end{aligned}$$

Janine has 3 nickels and 4 dimes.

Sometimes a question asks you to determine (from concentrations) how much of a particular substance to use. The substance in question could be something like coins as above, or it could be a chemical in solution, or even heat. In such a case, you need to know the amount of whatever substance is in each part. There are several common situations where to get one equation you simply add two given quantities, but to get the second equation you need to use a **product.** Three examples are below.

TABLE 12.5:

Type of mixture	First equation	Second equation
Coins (items with $ value)	total number of items (n_1+n_2)	total ***value*** (item value $\times$ no. of items)
Chemical solutions	total solution volume (V_1+V_2)	amount of ***solute*** (vol $\times$ concentration)
Density of two substances	total amount or volume of mix	total ***mass*** (volume $\times$ density)

For example, when considering mixing chemical solutions, we will most likely need to consider the total amount of *solute* in the individual parts and in the final mixture. (A solute is the chemical that is dissolved in a solution. An example of a solute is salt when added to water to make a brine.) To find the total amount, simply multiply the amount of the mixture by the **fractional concentration**. To illustrate, lets look at an example where you are given amounts relative to the whole.

Example 5

A chemist needs to prepare 500 ml of copper-sulfate solution with a 15% concentration. She wishes to use a high concentration solution (60%) and dilute it with a low concentration solution (5%) in order to do this. How much of each solution should she use?

Solution

To set this problem up, we first need to define our variables. Our unknowns are the amount of concentrated solution (x) and the amount of dilute solution (y). We will also convert the percentages (60%, 15% and 5%) into decimals

(0.6, 0.15 and 0.05). The two pieces of critical information are the final volume (500 ml) and the final amount of solute (15% of 500 $ml = 75$ ml). Our equations will look like this:

Volume equation: $x + y = 500$

Solute equation: $0.6x + 0.05y = 75$

To isolate a variable for substitution, we can see its easier to start with equation 1:

$$\begin{aligned}
&x + y = 500 && \textit{subtract } y \textit{ from both sides}:\\
&x = 500 - y && \textit{now substitute into equation } 2:\\
&0.6(500 - y) + 0.05y = 75 && \textit{distribute the } 0.6:\\
&300 - 0.6y + 0.05y = 75 && \textit{collect like terms}:\\
&300 - 0.55y = 75 && \textit{subtract } 300 \textit{ from both sides}:\\
&-0.55y = -225 && \textit{divide both sides by } -0.55:\\
&\underline{y = 409\ ml} && \textit{substitute back into equation for } x:\\
&x = 500 - 409 = \underline{91\ ml} &&
\end{aligned}$$

So the chemist should mix 91 ml of the 60% solution with 409 ml of the 5% solution.

Further Practice

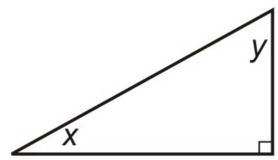
For lots more practice solving linear systems, check out this web page: http://www.algebra.com/algebra/homework/coordinate/practice-linear-system.epl

After clicking to see the solution to a problem, you can click the back button and then click Try Another Practice Linear System to see another problem.

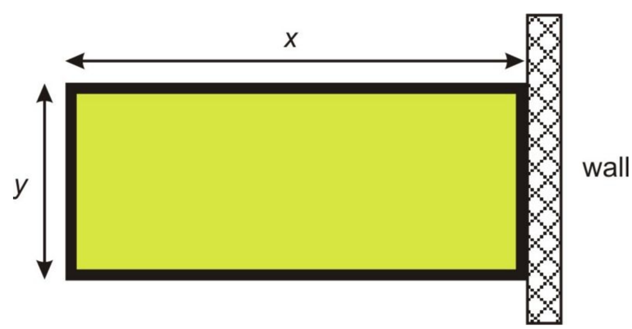
Practice Set

1. Solve the system:
   $x + 2y = 9$
   $3x + 5y = 20$
2. Solve the system:
   $x - 3y = 10$
   $2x + y = 13$
3. Solve the system:
   $2x + 0.5y = -10$
   $x - y = -10$
4. Solve the system:
   $2x + 0.5y = 3$
   $x + 2y = 8.5$
5. Solve the system:
   $3x + 5y = -1$
   $x + 2y = -1$

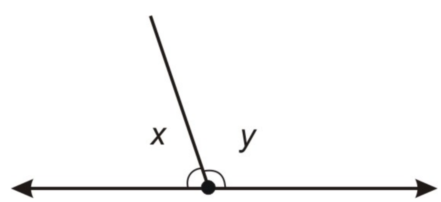
6. Solve the system:
$3x+5y=-3$
$x+2y=-\frac{4}{3}$
7. Solve the system:
$x-y=-\frac{12}{5}$
$2x+5y=-2$
8. Of the two non-right angles in a right angled triangle, one measures twice as many degrees as the other. What are the angles?



9. The sum of two numbers is 70. They differ by 11. What are the numbers?
10. A number plus half of another number equals 6; twice the first number minus three times the second number equals 4. What are the numbers?
11. A rectangular field is enclosed by a fence on three sides and a wall on the fourth side. The total length of the fence is 320 yards. If the field has a total perimeter of 400 yards, what are the dimensions of the field?



12. A ray cuts a line forming two angles. The difference between the two angles is 18°. What does each angle measure?



13. I have $15 and wish to buy five pounds of mixed nuts for a party. Peanuts cost $2.20 per pound. Cashews cost $4.70 per pound.
    (a) How many pounds of each should I buy?
    (b) If I suddenly realize I need to set aside $5 to buy chips, can I still buy 5 pounds of nuts with the remaining $10?
    (c) Whats the greatest amount of nuts I can buy?
14. A chemistry experiment calls for one liter of sulfuric acid at a 15% concentration, but the supply room only stocks sulfuric acid in concentrations of 10% and 35%.
    (a) How many liters of each should be mixed to give the acid needed for the experiment?
    (b) How many liters should be mixed to give *two* liters at a 15% concentration?

15. Bachelle wants to know the density of her bracelet, which is a mix of gold and silver. Density is total mass divided by total volume. The density of gold is 19.3 g/cc and the density of silver is 10.5 g/cc. The jeweler told her that the volume of silver in the bracelet was 10 cc and the volume of gold was 20 cc. Find the combined density of her bracelet.
16. Jason is five years older than Becky, and the sum of their ages is 23. What are their ages?
17. Tickets to a show cost $10 in advance and $15 at the door. If 120 tickets are sold for a total of $1390, how many of the tickets were bought in advance?
18. The multiple-choice questions on a test are worth 2 points each, and the short-answer questions are worth 5 points each.
    (a) If the whole test is worth 100 points and has 35 questions, how many of the questions are multiple-choice and how many are short-answer?
    (b) If Kwan gets 31 questions right and ends up with a score of 86 on the test, how many questions of each type did she get right? (Assume there is no partial credit.)
    (c) If Ashok gets 5 questions wrong and ends up with a score of 87 on the test, how many questions of each type did he get wrong? (Careful!)
    (d) What are two ways you could have set up the equations for part c?
    (e) How could you have set up part b differently?

12.3 Solving Linear Systems by Elimination

Introduction

In this lesson, well see how to use simple addition and subtraction to simplify our system of equations to a single equation involving a single variable. Because we go from two unknowns (x and y) to a single unknown (either x or y), this method is often referred to by ***solving by elimination***. We eliminate one variable in order to make our equations solvable! To illustrate this idea, lets look at the simple example of buying apples and bananas.

Example 1

If one apple plus one banana costs $1.25 and one apple plus 2 bananas costs $2.00, how much does one banana cost? One apple?

It shouldnt take too long to discover that each banana costs $0.75. After all, the second purchase just contains 1 more banana than the first, and costs $0.75 more, so that one banana must cost $0.75.

Heres what we get when we describe this situation with algebra:

$$\begin{aligned} a+b &= 1.25 \\ a+2b &= 2.00 \end{aligned}$$

Now we can subtract the number of apples and bananas in the first equation from the number in the second equation, and also subtract the cost in the first equation from the cost in the second equation, to get the *difference* in cost that corresponds to the *difference* in items purchased.

$$(a+2b)-(a+b) = 2.00-1.25 \rightarrow b = 0.75$$

That gives us the cost of one banana. To find out how much one apple costs, we subtract $0.75 from the total cost of one apple and one banana.

$$a+0.75 = 1.25 \rightarrow a = 1.25-0.75 \rightarrow a = 0.50$$

So an apple costs 50 cents.

To solve systems using addition and subtraction, well be using exactly this idea by looking at the ***sum*** or ***difference*** of the two equations we can determine a value for one of the unknowns.

Solving Linear Systems Using Addition of Equations

Often considered the easiest method of solving systems of equations, the addition (or elimination) method lets us combine two equations in such a way that the resulting equation has only one variable. We can then use simple algebra to solve for that variable. Then, if we need to, we can substitute the value we get for that variable back into either one of the original equations to solve for the other variable.

Example 2

Solve this system by addition:

$$\begin{aligned}3x+2y&=11\\5x-2y&=13\end{aligned}$$

Solution

We will add **everything** on the left of the equals sign from both equations, and this will be equal to the sum of everything on the right:

$$(3x+2y)+(5x-2y)=11+13 \rightarrow 8x=24 \rightarrow x=3$$

A simpler way to visualize this is to keep the equations as they appear above, and to add them together vertically, going down the columns. However, just like when you add units, tens and hundreds, you MUST be sure to keep the x's and y's in their own columns. You may also wish to use terms like "$0y$" as a placeholder!

$$\begin{array}{r}3x+2y=11\\ \underline{+\;\;(5x-2y)=13}\\ 8x+0y=24\end{array}$$

Again we get $8x=24$, or $x=3$. To find a value for y, we simply substitute our value for x back in.

Substitute $x=3$ into the second equation:

$$\begin{aligned}5\cdot 3-2y&=13 && since\ 5\times 3=15,\ we\ subtract\ 15\ from\ both\ sides:\\-2y&=-2 && divide\ by\ -2\ to\ get:\\y&=1\end{aligned}$$

The reason this method worked is that the $y-$coefficients of the two equations were opposites of each other: 2 and -2. Because they were opposites, they canceled each other out when we added the two equations together, so our final equation had no $y-$term in it and we could just solve it for x.

In a little while well see how to use the addition method when the coefficients are not opposites, but for now lets look at another example where they are.

Example 3

Andrew is paddling his canoe down a fast-moving river. Paddling downstream he travels at 7 miles per hour, relative to the river bank. Paddling upstream, he moves slower, traveling at 1.5 miles per hour. If he paddles equally hard in both directions, how fast is the current? How fast would Andrew travel in calm water?

Solution

First we convert our problem into equations. We have two unknowns to solve for, so well call the speed that Andrew paddles at x, and the speed of the river y. When traveling downstream, Andrew speed is boosted by the river current, so his total speed is his paddling speed *plus* the speed of the river $(x+y)$. Traveling upstream, the river is working against him, so his total speed is his paddling speed *minus* the speed of the river $(x-y)$.

Downstream Equation: $x+y=7$

Upstream Equation: $x - y = 1.5$

Next well eliminate one of the variables. If you look at the two equations, you can see that the coefficient of y is +1 in the first equation and -1 in the second. Clearly $(+1) + (-1) = 0$, so this is the variable we will eliminate. To do this we simply add equation 1 to equation 2. We must be careful to collect like terms, and make sure that everything on the left of the equals sign stays on the left, and everything on the right stays on the right:

$$(x+y)+(x-y) = 7+1.5 \Rightarrow 2x = 8.5 \Rightarrow x = 4.25$$

Or, using the column method we used in example 2:

$$\begin{array}{r} x+y=7 \\ + \;\; x-y=1.5 \\ \hline 2x+0y=8.5 \end{array}$$

Again we get $2x = 8.5$, or $x = 4.25$. To find a corresponding value for y, we plug our value for x into either equation and isolate our unknown. In this example, well plug it into the first equation:

$$\begin{aligned} 4.25+y &= 7 && \text{subtract } 4.25 \text{ from both sides}: \\ y &= 2.75 \end{aligned}$$

Andrew paddles at 4.25 miles per hour. The river moves at 2.75 miles per hour.

Solving Linear Systems Using Multiplication

So far, weve seen that the elimination method works well when the coefficient of one variable happens to be the same (or opposite) in the two equations. But what if the two equations dont have any coefficients the same?

It turns out that we can still use the elimination method; we just have to *make* one of the coefficients match. We can accomplish this by multiplying one or both of the equations by a constant.

Heres a quick review of how to do that. Consider the following questions:

1. If 10 apples cost \$5, how much would 30 apples cost?
2. If 3 bananas plus 2 carrots cost \$4, how mush would 6 bananas plus 4 carrots cost?

If you look at the first equation, it should be obvious that each apple costs \$0.50. So 30 apples should cost \$15.00.

The second equation is trickier; it isnt obvious what the individual price for either bananas or carrots is. Yet we know that the answer to question 2 is \$8.00. How?

If we look again at question 1, we see that we can write an equation: $10a = 5$ (a being the cost of 1 apple). So to find the cost of 30 apples, we *could* solve for a and then multiply by 30but we could also just multiply both sides of the equation by 3. We would get $30a = 15$, and that tells us that 30 apples cost \$15.

And we can do the same thing with the second question. The equation for this situation is $3b + 2c = 4$, and we can see that we need to solve for $(6b + 4c)$, which is simply 2 times $(3b + 2c)$! So algebraically, we are simply multiplying the entire equation by 2:

$$\begin{aligned} 2(3b+2c) &= 2 \cdot 4 && \text{distribute and multiply}: \\ 6b+4c &= 8 \end{aligned}$$

So when we multiply an equation, all we are doing is multiplying every term in the equation by a fixed amount.

Solving a Linear System by Multiplying One Equation

If we can multiply every term in an equation by a fixed number (a **scalar**), that means we can use the addition method on a whole new set of linear systems. We can manipulate the equations in a system to ensure that the coefficients of one of the variables match.

This is easiest to do when the coefficient as a variable in one equation is a multiple of the coefficient in the other equation.

Example 4

Solve the system:

$$7x+4y=17$$
$$5x-2y=11$$

Solution

You can easily see that if we multiply the second equation by 2, the coefficients of y will be +4 and -4, allowing us to solve the system by addition:

2 times equation 2:

$$\begin{array}{rl} 10x-4y=22 & \qquad\qquad \textit{now add to equation one}: \\ +\ \ (7x+4y)=17 & \\ \hline 17x=34 & \end{array}$$

$$\textit{divide by } 17 \textit{ to get}: \qquad x=2$$

Now simply substitute this value for x back into equation 1:

$$\begin{aligned} 7\cdot 2+4y&=17 && \textit{since } 7\times 2=14, \textit{ subtract } 14 \textit{ from both sides}: \\ 4y&=3 && \textit{divide by } 4: \\ y&=0.75 \end{aligned}$$

Example 5

Anne is rowing her boat along a river. Rowing downstream, it takes her 2 minutes to cover 400 yards. Rowing upstream, it takes her 8 minutes to travel the same 400 yards. If she was rowing equally hard in both directions, calculate, in yards per minute, the speed of the river and the speed Anne would travel in calm water.

Solution

Step one: first we convert our problem into equations. We know that *distance traveled* is equal to *speed* $\times$ *time*. We have two unknowns, so well call the speed of the river x, and the speed that Anne rows at y. When traveling downstream, her total speed is her rowing speed plus the speed of the river, or $(x+y)$. Going upstream, her speed is hindered by the speed of the river, so her speed upstream is $(x-y)$.

Downstream Equation: $2(x+y)=400$

Upstream Equation: $8(x-y)=400$

Distributing gives us the following system:

$$\begin{aligned} 2x+2y &= 400 \\ 8x-8y &= 400 \end{aligned}$$

Right now, we cant use the method of elimination because none of the coefficients match. But if we multiplied the top equation by 4, the coefficients of y would be +8 and -8. Lets do that:

$$\begin{aligned} 8x+8y &= 1,600 \\ + \quad (8x-8y) &= 400 \\ \hline 16x &= 2,000 \end{aligned}$$

Now we divide by 16 to obtain $x=125$.

Substitute this value back into the first equation:

$$\begin{aligned} 2(125+y) &= 400 && \textit{divide both sides by } 2: \\ 125+y &= 200 && \textit{subtract } 125 \textit{ from both sides}: \\ y &= 75 \end{aligned}$$

Anne rows at 125 yards per minute, and the river flows at 75 yards per minute.

Solving a Linear System by Multiplying Both Equations

So what do we do if none of the coefficients match and none of them are simple multiples of each other? We do the same thing we do when were adding fractions whose denominators arent simple multiples of each other. Remember that when we add fractions, we have to find a **lowest common denominator**that is, the lowest common multiple of the two denominatorsand sometimes we have to rewrite not just one, but both fractions to get them to have a common denominator. Similarly, sometimes we have to multiply both equations by different constants in order to get one of the coefficients to match.

Example 6

Andrew and Anne both use the I-Haul truck rental company to move their belongings from home to the dorm rooms on the University of Chicago campus. I-Haul has a charge per day and an additional charge per mile. Andrew travels from San Diego, California, a distance of 2060 miles in five days. Anne travels 880 miles from Norfolk, Virginia, and it takes her three days. If Anne pays $840 and Andrew pays $1845, what does I-Haul charge

a) *per day?*

b) *per mile traveled?*

Solution

First, well set up our equations. Again we have 2 unknowns: the **daily rate** (well call this x), and the **per-mile rate** (well call this y).

Annes equation: $3x+880y=840$

Andrews Equation: $5x+2060y=1845$

We cant just multiply a single equation by an integer number in order to arrive at matching coefficients. But if we look at the coefficients of x (as they are easier to deal with than the coefficients of y), we see that they both have a common multiple of 15 (in fact 15 is the ***lowest common multiple***). So we can multiply both equations.

Multiply the top equation by 5:

$$15x + 4400y = 4200$$

Multiply the lower equation by -3:

$$-15x - 6180y = -5535$$

Add:

$$\begin{array}{rr} & 15x + 4400y = 4200 \\ + & (-15x - 6180y) = -5535 \\ \hline & -1780y = -1335 \end{array}$$

$$\text{Divide by } -1780: \; y = 0.75$$

Substitute this back into the top equation:

$$\begin{aligned} 3x + 880(0.75) &= 840 && \textit{since } 880 \times 0.75 = 660, \textit{ subtract } 660 \textit{ from both sides}: \\ 3x &= 180 && \textit{divide both sides by } 3 \\ x &= 60 \end{aligned}$$

I-Haul charges \$60 per day plus \$0.75 per mile.

Comparing Methods for Solving Linear Systems

Now that weve covered the major methods for solving linear equations, lets review them. For simplicity, well look at them in table form. This should help you decide which method would be best for a given situation.

TABLE 12.6:

Method:	**Best used when you...**	**Advantages:**	**Comment:**
Graphing	...dont need an accurate answer.	Often easier to see number and quality of intersections on a graph. With a graphing calculator, it can be the fastest method since you dont have to do any computation.	Can lead to imprecise answers with non-integer solutions.

TABLE 12.6: (continued)

Method:	Best used when you...	Advantages:	Comment:
Substitution	...have an *explicit* equation for one variable (e.g. $y = 14x + 2$)	Works on all systems. Reduces the system to one variable, making it easier to solve.	You are not often given explicit functions in systems problems, so you may have to do extra work to get one of the equations into that form.
Elimination by Addition or Subtraction	...have matching coefficients for one variable in both equations.	Easy to combine equations to eliminate one variable. Quick to solve.	It is not very likely that a given system will have matching coefficients.
Elimination by Multiplication and then Addition and Subtraction	...do not have any variables defined explicitly or any matching coefficients.	Works on all systems. Makes it possible to combine equations to eliminate one variable.	Often more algebraic manipulation is needed to prepare the equations.

The table above is only a guide. You might prefer to use the graphical method for every system in order to better understand what is happening, or you might prefer to use the multiplication method even when a substitution would work just as well.

Example 7

Two angles are ***complementary*** *when the sum of their angles is* 90°. *Angles A and B are complementary angles, and twice the measure of angle A is* 9° *more than three times the measure of angle B. Find the measure of each angle.*

Solution

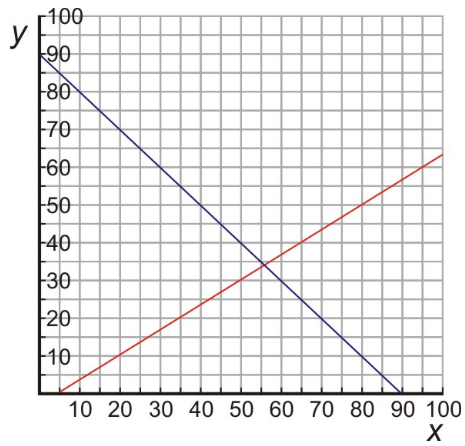
First we write out our 2 equations. We will use x to be the measure of angle A and y to be the measure of angle B. We get the following system:

$$\begin{aligned} x + y &= 90 \\ 2x &= 3y + 9 \end{aligned}$$

First, well solve this system with the graphical method. For this, we need to convert the two equations to $y = mx + b$ form:

$$\begin{aligned} x + y = 90 \qquad &\Rightarrow y = -x + 90 \\ 2x = 3y + 9 \qquad &\Rightarrow y = \frac{2}{3}x - 3 \end{aligned}$$

The first line has a slope of -1 and a $y-$intercept of 90, and the second line has a slope of $\frac{2}{3}$ and a $y-$intercept of -3. The graph looks like this:



In the graph, it appears that the lines cross at around $x = 55, y = 35$, but it is difficult to tell exactly! Graphing by hand is not the best method in this case!

Next, well try solving by substitution. Lets look again at the system:

$$x + y = 90$$
$$2x = 3y + 9$$

Weve already seen that we can start by solving either equation for y, so lets start with the first one:

$$y = 90 - x$$

Substitute into the second equation:

$$\begin{aligned}
2x &= 3(90 - x) + 9 && \textit{distribute the } 3: \\
2x &= 270 - 3x + 9 && \textit{add } 3x \textit{ to both sides}: \\
5x &= 270 + 9 = 279 && \textit{divide by } 5: \\
x &= 55.8^\circ
\end{aligned}$$

Substitute back into our expression for y:

$$y = 90 - 55.8 = 34.2^\circ$$

Angle A**measures** 55.8°; **angle** B**measures** 34.2°.

Finally, well try solving by elimination (with multiplication):

Rearrange equation one to standard form:

$$amp; x + y = 90 \qquad \Rightarrow 2x + 2y = 180$$

Multiply equation two by 2:

$$2x = 3y + 9 \qquad \Rightarrow 2x - 3y = 9$$

Subtract:

$$\begin{aligned} 2x + 2y &= 180 \\ - \quad (2x - 3y) &= -9 \\ \hline 5y &= 171 \end{aligned}$$

Divide by 5 to obtain $y = 34.2^\circ$

Substitute this value into the very first equation:

$$\begin{aligned} x + 34.2 &= 90 & \qquad & \textit{subtract } 34.2 \textit{ from both sides:} \\ x &= 55.8^\circ \end{aligned}$$

Angle A **measures** 55.8°; **angle** B **measures** 34.2°.

Even though this system looked ideal for substitution, the method of elimination worked well too. Once the equations were rearranged properly, the solution was quick to find. Youll need to decide yourself which method to use in each case you see from now on. Try to master all the techniques, and recognize which one will be most efficient for each system you are asked to solve.

The following Khan Academy video contains three examples of solving systems of equations using addition and subtraction as well as multiplication (which is the next topic): http://www.youtube.com/watch?v=nok99JOhcjo (9:57). (Note that the narrator is not always careful about showing his work, and you should try to be neater in your mathematical writing.)

For even more practice, we have this video. One common type of problem involving systems of equations (especially on standardized tests) is age problems." In the following video the narrator shows two examples of age problems, one involving a single person and one involving two people. Khan Academy Age Problems (7:13)

Practice Set

1. Solve the system:
   $3x + 4y = 2.5$
   $5x - 4y = 25.5$
2. Solve the system:
   $5x + 7y = -31$
   $5x - 9y = 17$

3. Solve the system:
   $3y-4x=-33$
   $5x-3y=40.5$
4. Nadia and Peter visit the candy store. Nadia buys three candy bars and four fruit roll-ups for $2.84. Peter also buys three candy bars, but can only afford one additional fruit roll-up. His purchase costs $1.79. What is the cost of a candy bar and a fruit roll-up individually?
5. A small plane flies from Los Angeles to Denver with a tail wind (the wind blows in the same direction as the plane) and an air-traffic controller reads its ground-speed (speed measured relative to the ground) at 275 miles per hour. Another, identical plane, moving in the opposite direction has a ground-speed of 227 miles per hour. Assuming both planes are flying with identical air-speeds, calculate the speed of the wind.
6. An airport taxi firm charges a pick-up fee, plus an additional per-mile fee for any rides taken. If a 12-mile journey costs $14.29 and a 17-mile journey costs $19.91, calculate:
   (a) the pick-up fee
   (b) the per-mile rate
   (c) the cost of a seven mile trip
7. Calls from a call-box are charged per minute at one rate for the first five minutes, then a different rate for each additional minute. If a 7-minute call costs $4.25 and a 12-minute call costs $5.50, find each rate.
8. A plumber and a builder were employed to fit a new bath, each working a different number of hours. The plumber earns $35 per hour, and the builder earns $28 per hour. Together they were paid $330.75, but the plumber earned $106.75 more than the builder. How many hours did each work?
9. Paul has a part time job selling computers at a local electronics store. He earns a fixed hourly wage, but can earn a bonus by selling warranties for the computers he sells. He works 20 hours per week. In his first week, he sold eight warranties and earned $220. In his second week, he managed to sell 13 warranties and earned $280. What is Pauls hourly rate, and how much extra does he get for selling each warranty?

Solve the following systems using multiplication.

10. $5x-10y=15$
    $3x-2y=3$
11. $5x-y=10$
    $3x-2y=-1$
12. $5x+7y=15$
    $7x-3y=5$
13. $9x+5y=9$
    $12x+8y=12.8$
14. $4x-3y=1$
    $3x-4y=4$
15. $7x-3y=-3$
    $6x+4y=3$

Solve the following systems using any method.

16. $x=3y$
    $x-2y=-3$
17. $y=3x+2$
    $y=-2x+7$
18. $5x-5y=5$
    $5x+5y=35$
19. $y=-3x-3$
    $3x-2y+12=0$

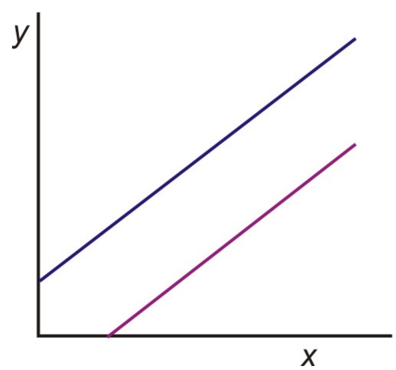
20. $3x - 4y = 3$
    $4y + 5x = 10$
21. $9x - 2y = -4$
    $2x - 6y = 1$
22. Supplementary angles are two angles whose sum is 180°. Angles A and B are supplementary angles. The measure of Angle A is 18° less than twice the measure of Angle B. Find the measure of each angle.
23. A farmer has fertilizer in 5% and 15% solutions. How much of each type should he mix to obtain 100 liters of fertilizer in a 12% solution?
24. A 150-yard pipe is cut to provide drainage for two fields. If the length of one piece is three yards less that twice the length of the second piece, what are the lengths of the two pieces?
25. Mr. Stein invested a total of $100,000 in two companies for a year. Company As stock showed a 13% annual gain, while Company B showed a 3% loss for the year. Mr. Stein made an 8% return on his investment over the year. How much money did he invest in each company?
26. A baker sells plain cakes for $7 and decorated cakes for $11. On a busy Saturday the baker started with 120 cakes, and sold all but three. His takings for the day were $991. How many plain cakes did he sell that day, and how many were decorated before they were sold?
27. Twice Johns age plus five times Claires age is 204. Nine times Johns age minus three times Claires age is also 204. How old are John and Claire?

12.4 Special Types of Linear Systems

Introduction

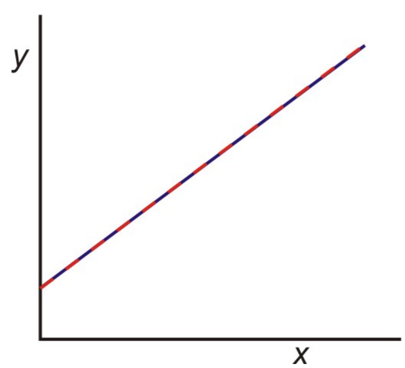
As we saw previously, a system of linear equations is a set of linear equations which must be solved together. The lines in the system can be graphed together on the same coordinate graph and the solution to the system is the point at which the two lines intersect.

Or at least thats what usually happens. But what if the lines turn out to be parallel when we graph them?



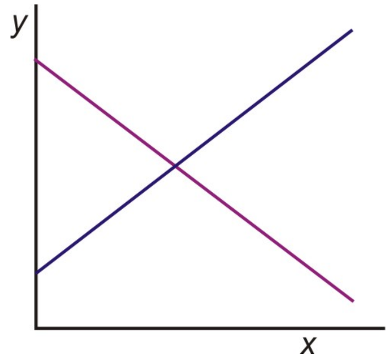
If the lines are parallel, they wont ever intersect. That means that the system of equations they represent has no solution. A system with no solutions is called an **inconsistent system.**

And what if the lines turn out to be identical?



If the two lines are the same, then *every* point on one line is also on the other line, so every point on the line is a solution to the system. The system has an **infinite number** of solutions, and the two equations are really just different forms of the same equation. Such a system is called a **dependent system**.

But usually, two lines cross at exactly one point and the system has exactly one solution:



A system with exactly one solution is called a **consistent system**.

To identify a system as **consistent, inconsistent**, or **dependent**, we can graph the two lines on the same graph and see if they intersect, are parallel, or are the same line. But sometimes it is hard to tell whether two lines are parallel just by looking at a roughly sketched graph.

Another option is to write each line in slope-intercept form and compare the slopes and $y-$ intercepts of the two lines. To do this we must remember that:

- Lines with different slopes always intersect.
- Lines with the same slope but different $y-$intercepts are parallel.
- Lines with the same slope and the same $y-$intercepts are identical.

Example 1

Determine whether the following system has exactly one solution, no solutions, or an infinite number of solutions.

$$\begin{aligned} 2x - 5y &= 2 \\ 4x + y &= 5 \end{aligned}$$

Solution

We must rewrite the equations so they are in slope-intercept form

$$\begin{array}{lclcl} 2x - 5y = 2 & & -5y = -2x + 2 & & y = \frac{2}{5}x - \frac{2}{5} \\ & \Rightarrow & & \Rightarrow & \\ x + y = 5 & & y = -4x + 5 & & y = -4x + 5 \end{array}$$

The slopes of the two equations are different; therefore the lines must cross at a single point and the system has exactly one solution. This is a **consistent system.**

Example 2

Determine whether the following system has exactly one solution, no solutions, or an infinite number of solutions.

$$\begin{aligned} 3x &= 5 - 4y \\ 6x + 8y &= 7 \end{aligned}$$

Solution

We must rewrite the equations so they are in slope-intercept form

$$\begin{array}{lclcl} 3x = 5 - 4y & & 4y = -3x + 5 & & y = -\frac{3}{4}x + \frac{5}{4} \\ & \Rightarrow & & \Rightarrow & \\ 6x + 8y = 7 & & 8y = -6x + 7 & & y = -\frac{3}{4}x + \frac{7}{8} \end{array}$$

The slopes of the two equations are the same but the $y-$intercepts are different; therefore the lines are parallel and the system has no solutions. This is an **inconsistent system.**

Example 3

Determine whether the following system has exactly one solution, no solutions, or an infinite number of solutions.

$$\begin{aligned} x+y&=3 \\ 3x+3y&=9 \end{aligned}$$

Solution

We must rewrite the equations so they are in slope-intercept form

$$\begin{array}{lclcl} x+y=3 & & y=-x+3 & & y=-x+3 \\ & \Rightarrow & & \Rightarrow & \\ x+3y=9 & & 3y=-3x+9 & & y=-x+3 \end{array}$$

The lines are identical; therefore the system has an infinite number of solutions. It is a **dependent system.**

Determining the Type of System Algebraically

A third option for identifying systems as consistent, inconsistent or dependent is to just solve the system and use the result as a guide.

Example 4

Solve the following system of equations. Identify the system as consistent, inconsistent or dependent.

$$\begin{aligned} 10x-3y&=3 \\ 2x+y&=9 \end{aligned}$$

Solution

Lets solve this system using the substitution method.

Solve the second equation for y:

$$2x+y=9 \Rightarrow y=-2x+9$$

Substitute that expression for y in the first equation:

$$\begin{aligned} 10x-3y&=3 \\ 10x-3(-2x+9)&=3 \\ 10x+6x-27&=3 \\ 16x&=30 \\ x&=\frac{15}{8} \end{aligned}$$

Substitute the value of x back into the second equation and solve for y:

$$2x+y=9 \Rightarrow y=-2x+9 \Rightarrow y=-2\cdot\frac{15}{8}+9 \Rightarrow y=\frac{21}{4}$$

The solution to the system is $\left(\frac{15}{8},\frac{21}{4}\right)$. The system is **consistent** since it has only one solution.

Example 5

Solve the following system of equations. Identify the system as consistent, inconsistent or dependent.

$$\begin{aligned}3x-2y&=4\\9x-6y&=1\end{aligned}$$

Solution

Lets solve this system by the method of multiplication.

Multiply the first equation by 3:

$$\begin{array}{lcl}3(3x-2y=4) & & 9x-6y=12\\ & \Rightarrow & \\ 9x-6y=1 & & 9x-6y=1\end{array}$$

Add the two equations:

$$\begin{aligned}9x-6y&=4\\\underline{9x-6y}&\underline{=1}\\0&=13 \quad \text{This statement is not true.}\end{aligned}$$

If our solution to a system turns out to be a statement that is not true, then the system doesnt really have a solution; it is **inconsistent.**

Example 6

Solve the following system of equations. Identify the system as consistent, inconsistent or dependent.

$$\begin{aligned}4x+y&=3\\12x+3y&=9\end{aligned}$$

Solution

Lets solve this system by substitution.

Solve the first equation for y:

$$4x+y=3 \Rightarrow y=-4x+3$$

Substitute this expression for y in the second equation:

$$\begin{aligned}12x+3y&=9\\12x+3(-4x+3)&=9\\12x-12x+9&=9\\9&=9\end{aligned}$$

This statement is always true.

If our solution to a system turns out to be a statement that is always true, then the system is **dependent.**

A second glance at the system in this example reveals that the second equation is three times the first equation, so the two lines are identical. The system has an infinite number of solutions because they are really the same equation and trace out the same line.

Lets clarify this statement. An infinite number of solutions does not mean that *any* ordered pair (x,y) satisfies the system of equations. Only ordered pairs that solve the equation in the system (either one of the equations) are also solutions to the system. There are infinitely many of these solutions to the system because there are infinitely many points on any one line.

For example, (1, -1) is a solution to the system in this example, and so is (-1, 7). Each of them fits both the equations because both equations are really the same equation. But (3, 5) doesnt fit either equation and is not a solution to the system.

In fact, for every $x-$value there is just one $y-$value that fits both equations, and for every $y-$value there is exactly one $x-$valuejust as there is for a single line.

Lets summarize how to determine the type of system we are dealing with algebraically.

- A **consistent system** will always give exactly one solution.
- An **inconsistent system** will yield a statement that is *always false* (like $0 = 13$).
- A **dependent system** will yield a statement that is *always true* (like $9 = 9$).

Applications

In this section, well see how consistent, inconsistent and dependent systems might arise in real life.

Example 7

The movie rental store CineStar offers customers two choices. Customers can pay a yearly membership of $45 and then rent each movie for $2 or they can choose not to pay the membership fee and rent each movie for $3.50. How many movies would you have to rent before the membership becomes the cheaper option?

Solution

Lets translate this problem into algebra. Since there are two different options to consider, we can write two different equations and form a system.

The choices are membership and no membership. Well call the number of movies you rent x and the total cost of renting movies for a year y.

TABLE 12.7:

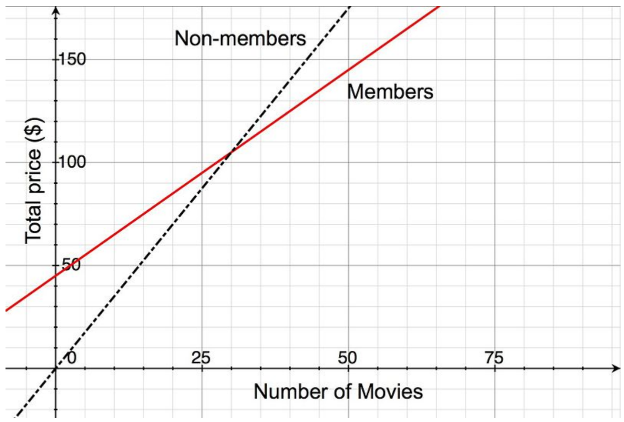
	flat fee	**rental fee**	**total**
membership	$45	$2x$	$y = 45 + 2x$
no membership	$0	$3.50x$	$y = 3.5x$

The flat fee is the dollar amount you pay per year and the rental fee is the dollar amount you pay when you rent a movie. For the membership option the rental fee is $2x$, since you would pay $2 for each movie you rented; for the no membership option the rental fee is $3.50x$, since you would pay $3.50 for each movie you rented.

Our system of equations is:

$$y = 45 + 2x$$
$$y = 3.50x$$

Heres a graph of the system:



Now we need to find the exact intersection point. Since each equation is already solved for y, we can easily solve the system with substitution. Substitute the second equation into the first one:

$$\begin{array}{l} y = 45 + 2x \\ \qquad \Rightarrow 3.50x = 45 + 2x \Rightarrow 1.50x = 45 \Rightarrow x = 30 \text{ movies} \\ y = 3.50x \end{array}$$

You would have to rent **30 movies per year** before the membership becomes the better option.

This example shows a real situation where a consistent system of equations is useful in finding a solution. Remember that for a consistent system, the lines that make up the system intersect at single point. In other words, the lines are not parallel or the slopes are different.

In this case, the slopes of the lines represent the price of a rental per movie. The lines cross because the price of rental per movie is different for the two options in the problem

Now lets look at a situation where the system is inconsistent. From the previous explanation, we can conclude that the lines will not intersect if the slopes are the same (and the $y-$intercept is different). Lets change the previous problem so that this is the case.

Example 8

Two movie rental stores are in competition. Movie House charges an annual membership of $30 and charges $3 per movie rental. Flicks for Cheap charges an annual membership of $15 and charges $3 per movie rental. After how many movie rentals would Movie House become the better option?

Solution

It should already be clear to see that Movie House will never become the better option, since its membership is more expensive and it charges the same amount per movie as Flicks for Cheap.

The lines on a graph that describe each option have different $y-$interceptsnamely 30 for Movie House and 15 for Flicks for Cheapbut the same slope: 3 dollars per movie. This means that the lines are parallel and so the system is inconsistent.

Now lets see how this works algebraically. Once again, well call the number of movies you rent x and the total cost of renting movies for a year y.

TABLE 12.8:

	flat fee	rental fee	total
Movie House	$30	$3x$	$y = 30 + 3x$
Flicks for Cheap	$15	$3x$	$y = 15 + 3x$

The system of equations that describes this problem is:

$y = 30 + 3x$
$y = 15 + 3x$

Lets solve this system by substituting the second equation into the first equation:

$15 + 3x = 30 + 3x \Rightarrow 15 = 30$ This statement is always false.

This means that the system is **inconsistent.**

Example 9

Peter buys two apples and three bananas for $4. Nadia buys four apples and six bananas for $8 from the same store. How much does one banana and one apple costs?

Solution

We must write two equations: one for Peters purchase and one for Nadias purchase.

Lets say a is the cost of one apple and b is the cost of one banana.

TABLE 12.9:

	cost of apples	cost of bananas	total cost
Peter	$2a$	$3b$	$2a+3b=4$
Nadia	$4a$	$6b$	$4a+6b=8$

The system of equations that describes this problem is:

$2a + 3b = 4$
$4a + 6b = 8$

Lets solve this system by multiplying the first equation by -2 and adding the two equations:

$$\begin{array}{lcr} -2(2a+3b=4) & & -4a-6b=-8 \\ & \Rightarrow & \\ 4a+6b=8 & & \underline{4a+6b=8} \\ & & 0+0=0 \end{array}$$

This statement is always true. This means that the system is **dependent.**

Looking at the problem again, we can see that we were given exactly the same information in both statements. If Peter buys two apples and three bananas for $4, it makes sense that if Nadia buys twice as many apples (four apples) and twice as many bananas (six bananas) she will pay twice the price ($8). Since the second equation doesnt give us any new information, it doesnt make it possible to find out the price of each fruit.

Practice Set

Express each equation in slope-intercept form. Without graphing, state whether the system of equations is consistent, inconsistent or dependent.

1.

$$3x - 4y = 13$$
$$y = -3x - 7$$

2.

$$\frac{3}{5}x+y=3$$
$$1.2x+2y=6$$

3.

$$3x-4y=13$$
$$y=-3x-7$$

4.

$$3x-3y=3$$
$$x-y=1$$

5.

$$0.5x-y=30$$
$$0.5x-y=-30$$

6.

$$4x-2y=-2$$
$$3x+2y=-12$$

7. $3x+y=4$
   $y=5-3x$
8. $x-2y=7$
   $4y-2x=14$

Find the solution of each system of equations using the method of your choice. State if the system is inconsistent or dependent.

9. $3x+2y=4$
   $-2x+2y=24$
10. $5x-2y=3$
    $2x-3y=10$
11. $3x-4y=13$
    $y=-3x-7$
12. $5x-4y=1$
    $-10x+8y=-30$
13. $4x+5y=0$
    $3x=6y+4.5$
14. $-2y+4x=8$
    $y-2x=-4$
15. $x-\frac{1}{2}y=\frac{3}{2}$
    $3x+y=6$
16. $0.05x+0.25y=6$
    $x+y=24$
17. $x+\frac{2}{3}y=6$
    $3x+2y=2$
18. A movie theater charges $4.50 for children and $8.00 for adults.

(a) On a certain day, 1200 people enter the theater and \$8375 is collected. How many children and how many adults attended?
(b) The next day, the manager announces that she wants to see them take in \$10000 in tickets. If there are 240 seats in the house and only five movie showings planned that day, is it possible to meet that goal?
(c) At the same theater, a 16-ounce soda costs \$3 and a 32-ounce soda costs \$5. If the theater sells 12,480 ounces of soda for \$2100, how many people bought soda? (**Note:** Be careful in setting up this problem!)

19. Jamal placed two orders with an internet clothing store. The first order was for 13 ties and 4 pairs of suspenders, and totaled \$487. The second order was for 6 ties and 2 pairs of suspenders, and totaled \$232. The bill does not list the per-item price, but all ties have the same price and all suspenders have the same price. What is the cost of one tie and of one pair of suspenders?
20. An airplane took four hours to fly 2400 miles in the direction of the jet-stream. The return trip against the jet-stream took five hours. What were the airplanes speed in still air and the jet-stream's speed?
21. Nadia told Peter that she went to the farmers market and bought two apples and one banana, and that it cost her \$2.50. She thought that Peter might like some fruit, so she went back to the seller and bought four more apples and two more bananas. Peter thanked Nadia, but told her that he did not like bananas, so he would only pay her for four apples. Nadia told him that the second time she paid \$6.00 for the fruit.

(a) What did Peter find when he tried to figure out the price of four apples?
(b) Nadia then told Peter she had made a mistake, and she actually paid \$5.00 on her second trip. Now what answer did Peter get when he tried to figure out how much to pay her?
(c) Alicia then showed up and told them she had just bought 3 apples and 2 bananas from the same seller for \$4.25. Now how much should Peter pay Nadia for four apples?

CHAPTER 13 Problem Solving

Chapter Outline

This chapter is an appendix to the text. You will find a number of different problem solving activities that may or may not be used in the class. The problems can be great learning tools - so you may want to try them on your own!

13.1 A Problem-Solving Plan

Much of mathematics apply to real-world situations. To think critically and to problem solve are mathematical abilities. Although these capabilities may be the most challenging, they are also the most rewarding.

To be successful in applying mathematics in real-life situations, you must have a "toolbox" of strategies to assist you. The last few lessons of many chapters in this FlexBook are devoted to filling this toolbox so you to become a better problem solver and tackle mathematics in the real world.

Step #1: Read and Understand the Given Problem

Every problem you encounter gives you clues needed to solve it successfully. Here is a checklist you can use to help you understand the problem.

$\surd$ Read the problem carefully. Make sure you read all the sentences. Many mistakes have been made by failing to fully read the situation.

$\surd$ Underline or highlight key words. These include mathematical operations such as *sum, difference, product,* and mathematical verbs such as *equal, more than, less than, is.* Key words also include the nouns the situation is describing such as *time, distance, people,* etc.

$\surd$ Ask yourself if you have seen a problem like this before. Even though the nouns and verbs may be different, the general situation may be similar to something else you've seen.

$\surd$ What are you being asked to do? What is the question you are supposed to answer?

$\surd$ What facts are you given? These typically include numbers or other pieces of information.

Once you have discovered what the problem is about, the next step is to declare what variables will represent the nouns in the problem. Remember to use letters that make sense!

Step #2: Make a Plan to Solve the Problem

The next step in the problem-solving plan is to **make a plan** or **develop a strategy.** How can the information you know assist you in figuring out the unknown quantities?

Here are some common strategies that you will learn.

- Drawing a diagram
- Making a table
- Looking for a pattern
- Using guess and check
- Working backwards

- Using a formula
- Reading and making graphs
- Writing equations
- Using linear models
- Using dimensional analysis
- Using the right type of function for the situation

In most problems, you will use a combination of strategies. For example, drawing a diagram and looking for patterns are good strategies for most problems. Also, making a table and drawing a graph are often used together. The "writing an equation" strategy is the one you will work with the most frequently in your study of algebra.

Step #3: Solve the Problem and Check the Results

Once you develop a plan, you can use it to **solve the problem.**

The last step in solving any problem should always be to **check and interpret** the answer. Here are some questions to help you to do that.

- Does the answer make sense?
- If you substitute the solution into the original problem, does it make the sentence true?
- Can you use another method to arrive at the same answer?

Step #4: Compare Alternative Approaches

Sometimes a certain problem is best solved by using a specific method. Most of the time, however, it can be solved by using several different strategies. When you are familiar with all of the problem-solving strategies, it is up to you to choose the methods that you are most comfortable with and that make sense to you. In this book, we will often use more than one method to solve a problem. This way we can demonstrate the strengths and weaknesses of different strategies when applied to different types of problems.

Regardless of the strategy you are using, you should always implement the problem-solving plan when you are solving word problems. Here is a summary of the problem-solving plan.

Step 1: Understand the problem.

Step 2: Devise a plan – Translate. Come up with a way to solve the problem. Set up an equation, draw a diagram, make a chart, or construct a table as a start to begin your problem-solving plan.

Step 3: Carry out the plan – Solve.

Step 4: Check and Interpret: Check to see if you have used all your information. Then look to see if the answer makes sense.

Solve Real-World Problems Using a Plan

Example 1: Jeff is 10 years old. His younger brother, Ben, is 4 years old. How old will Jeff be when he is twice as old as Ben?

Solution: Begin by understanding the problem. Highlight the key words.

Jeff is **10** years old. His younger brother, **Ben**, is **4** years old. **How old** will Jeff be **when he is twice as old as Ben**?

The question we need to answer is. "What is Jeff's age when he is twice as old as Ben?"

You could guess and check, use a formula, make a table, or look for a pattern.

The key is "twice as old." This clue means two times, or double Ben's age. Begin by doubling possible ages. Let's look for a pattern.

$4 \times 2 = 8$. Jeff is already older than 8.

$5 \times 2 = 10$. This doesn't make sense because Jeff is already 10.

$6 \times 2 = 12$. In two years, Jeff will be 12 and Ben will be 6. Jeff will be twice as old.

Jeff will be 12 years old.

Example 2: Matthew is planning to harvest his corn crop this fall. The field has 660 rows of corn with 300 ears per row. Matthew estimates his crew will have the crop harvested in 20 hours. How many ears of corn will his crew harvest per hour?

Solution: Begin by highlighting the key information.

Matthew is planning to harvest his corn crop this fall. The field has **660 rows** of corn with **300 ears per row**. Matthew estimates his crew will have the **crop harvested in 20 hours. How many ears of corn** will his crew **harvest per hour**?

You could draw a picture (it may take a while), write an equation, look for a pattern, or make a table. Let's try to use reasoning.

We need to figure out how many ears of corn are in the field. $660(300) = 198,000$. This is how many ears are in the field. It will take 20 hours to harvest the entire field, so we need to divide 198,000 by 20 to get the number of ears picked per hour.

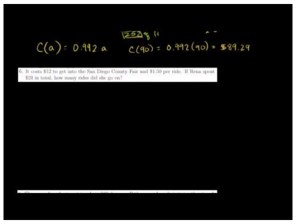
$$\frac{198,000}{20} = 9,900$$

The crew can harvest 9,900 ears per hour.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the

practice exercise listed in the following exercise set. However, the practice exercise is the same in both. CK-12 Basic Algebra:Word Problem-Solving Plan 1 (10:12)



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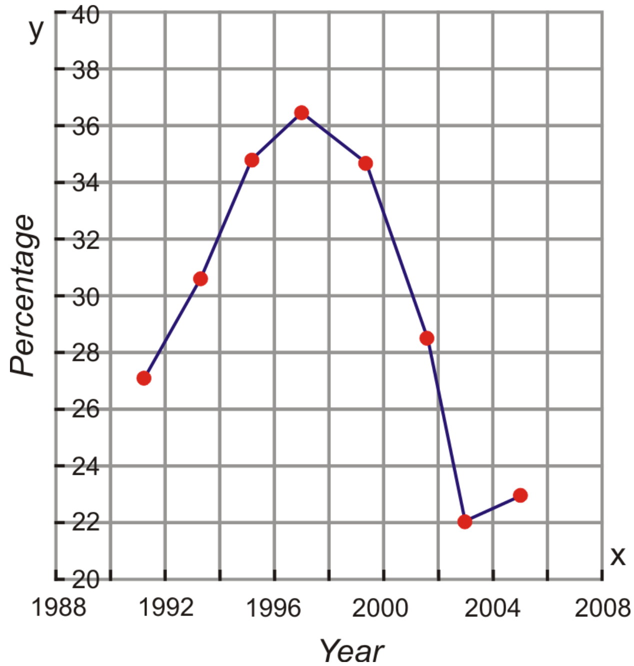
1. What are the four steps to solving a problem?
2. Name three strategies you can use to help make a plan. Which one(s) are you most familiar with already?
3. Which types of strategies work well together? Why?
4. Suppose Matthew's crew takes 36 hours to harvest the field. How many ears per hour will they harvest?
5. Why is it difficult to solve Ben and Jeff's age problem by drawing a diagram?
6. How do you check a solution to a problem? What is the purpose of checking the solution?
7. There were 12 people on a jury, with four more women than men. How many women were there?
8. A rope 14 feet long is cut into two pieces. One piece is 2.25 feet longer than the other. What are the lengths of the two pieces?
9. A sweatshirt costs \$35. Find the total cost if the sales tax is 7.75%.
10. This year you got a 5% raise. If your new salary is \$45,000, what was your salary before the raise?
11. It costs \$250 to carpet a room that is $14\ ft \times 18\ ft$. How much does it cost to carpet a room that is $9\ ft \times 10\ ft$?
12. A department store has a 15% discount for employees. Suppose an employee has a coupon worth \$10 off any item and she wants to buy a \$65 purse. What is the final cost of the purse if the employee discount is applied before the coupon is subtracted?
13. To host a dance at a hotel, you must pay \$250 plus \$20 per guest. How much money would you have to pay for 25 guests?
14. It costs \$12 to get into the San Diego County Fair and \$1.50 per ride. If Rena spent \$24 in total, how many rides did she go on?
15. An ice cream shop sells a small cone for \$2.92, a medium cone for \$3.50, and a large cone for \$4.25. Last Saturday, the shop sold 22 small cones, 26 medium cones, and 15 large cones. How much money did the store earn?
16. The sum of angles in a triangle is 180 degrees. If the second angle is twice the size of the first angle and the third angle is three times the size of the first angle, what are the measures of the angles in the triangle?

Mixed Review

17. Choose an appropriate variable for the following situation: *It takes Lily 45 minutes to bathe and groom a dog. How many dogs can she groom in an 9-hour day?*
18. Translate the following into an algebraic inequality: *Fourteen less than twice a number is greater than or equal to 16.*
19. Write the pattern of the table below in words and using an algebraic equation.

$amp;x$	-2	-1	0	1
$amp;y$	-8	-4	0	4

20. Check that $m = 4$ is a solution to $3y - 11 \geq -3$.
21. What is the domain and range of the graph below?


y
40
38
36
34
32
30
28
26
24
22
20
Percentage
x
1988
1992
1996
2000
2004
2008
Year

13.2 Problem-Solving Strategies: Make a Table; Look for a Pattern

This lesson focuses on two of the strategies introduced in the previous chapter: making a table and looking for a pattern. These are the most common strategies you have used before algebra. Let's review the four-step problem-solving plan from Lesson 1.7.

Step 1: Understand the problem.

Step 2: Devise a plan – Translate. Come up with a way to solve the problem. Set up an equation, draw a diagram, make a chart, or construct a table as a start to begin your problem-solving plan.

Step 3: Carry out the plan – Solve.

Step 4: Check and Interpret: Check to see if you used all your information. Then look to see if the answer makes sense.

Using a Table to Solve a Problem

When a problem has data that needs to be organized, a table is a highly effective problem-solving strategy. A table is also helpful when the problem asks you to record a large amount of information. Patterns and numerical relationships are easier to see when data are organized in a table.

Example 1: *Josie takes up jogging. In the first week she jogs for 10 minutes per day, in the second week she jogs for 12 minutes per day. Each week, she wants to increase her jogging time by 2 minutes per day. If she jogs six days per week each week, what will be her total jogging time in the sixth week?*

Solution: Organize the information in a table

TABLE 13.1:

Week 1	Week 2	Week 3	Week 4
10 minutes	12 minutes	14 minutes	16 minutes
60 min/week	72 min/week	84 min/week	96 min/week

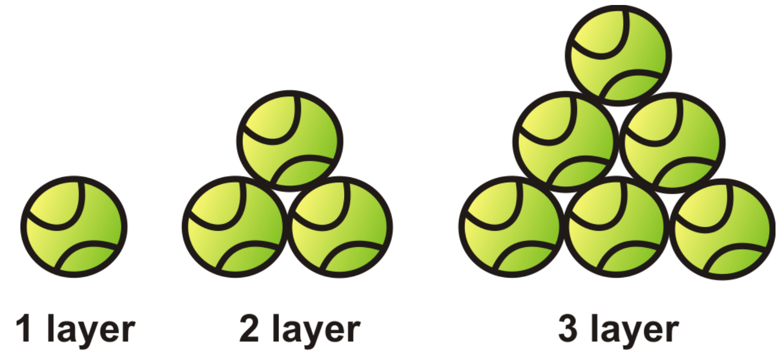
We can see the pattern that the number of minutes is increasing by 12 each week. Continuing this pattern, Josie will run 120 minutes in the sixth week.

Don't forget to check the solution! The pattern starts at 60 and adds 12 each week after the first week. The equation to represent this situation is $t = 60 + 12(w - 1)$. By substituting 6 for the variable of w, the equation becomes $t = 60 + 12(6 - 1) = 60 + 60 = 120$

Solve a Problem by Looking for a Pattern

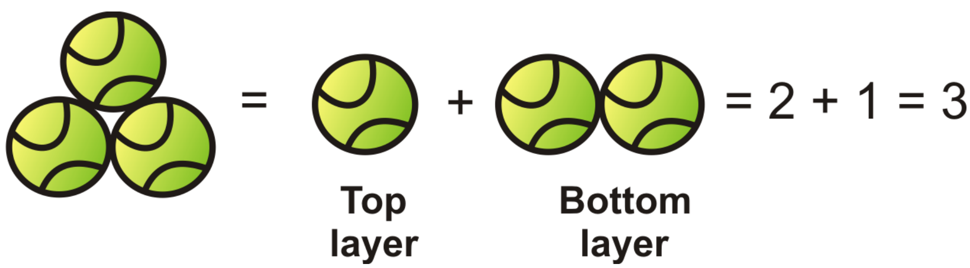
Some situations have a readily apparent pattern, which means that the pattern is easy to see. In this case, you may not need to organize the information into a table. Instead, you can use the pattern to arrive at your solution.

Example 2: *You arrange tennis balls in triangular shapes as shown. How many balls will there be in a triangle that has 8 layers?*

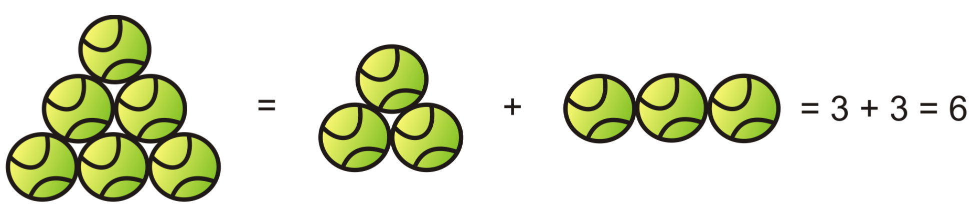


One layer: It is simple to see that a triangle with one layer has only one ball.

Two layers: For a triangle with two layers we add the balls from the top layer to the balls of the bottom layer. It is useful to make a sketch of the different layers in the triangle.



Three layers: we add the balls from the top triangle to the balls from the bottom layer.



We can fill the first three rows of the table.

1	2	3	4
1	3	6	$6+4=10$

To find the number of tennis balls in 8 layers, continue the pattern.

5	6	7	8
$10+5=15$	$15+6=21$	$21+7=28$	$28+8=36$

There will be 36 tennis balls in the 8 layers.

Check: Each layer of the triangle has one more ball than the previous one. In a triangle with 8 layers, each layer has the smae number of balls as its position. When we add these we get:

$1+2+3+4+5+6+7+8 = 36$ balls

The answer checks out.

Comparing Alternative Approaches to Solving Problems

In this section, we will compare the methods of "Making a Table" and "Looking for a Pattern" by using each method in turn to solve a problem.

Example 3: *Andrew cashes a $180 check and wants the money in $10 and $20 bills. The bank teller gives him 12 bills. How many of each kind of bill does he receive?*

Solution: Method 1: Making a Table

Tens	0	2	4	6	8	10	12	14	16	18
Twenties	9	8	7	6	5	4	3	2	1	0

The combination that has a sum of 12 is six $10 bills and six $20 bills.

Method 2: Using a Pattern

The pattern is that for every pair of $10 bills, the number of $20 bills reduces by one. Begin with the most number of $20 bills. For every $20 bill lost, add two $10 bills.

$$6(\$10)+6(\$20) = \$180$$

Check: Six $10 bills and six $20 bills $= 6(\$10)+6(\$20) = \$60+\$120 = \$180$.

Using These Strategies to Solve Problems

Example 4: *Students are going to march in a homecoming parade. There will be one kindergartener, two first-graders, three second-graders, and so on through* 12^{th} *grade. How many students will be walking in the homecoming parade?*

Could you make a table? Absolutely. Could you look for a pattern? Absolutely.

Solution 1: Make a table:

$amp;K$	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	13

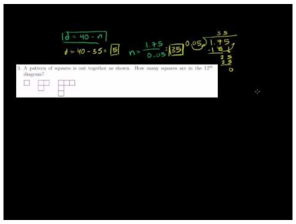
The solution is the sum of all the numbers, 91. There will be 91 students walking in the homecoming parade.

Solution 2: Look for a pattern.

The pattern is: The number of students is one more than their grade level. Therefore, the solution is the sum of numbers from 1 (kindergarten) through 13 (12^{th} grade). The solution is 91.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. CK-12 Basic Algebra:Word Problem-Solving Strategies (12:51)



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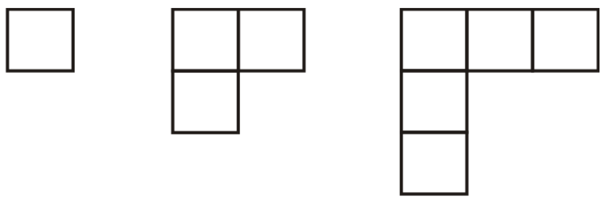
1. Go back and find the solution to the problem in Example 1.
2. Britt has \$2.25 in nickels and dimes. If she has 40 coins in total how many of each coin does she have?
3. A pattern of squares is placed together as shown. How many squares are in the 12^{th} diagram?



4. Oswald is trying to cut down on drinking coffee. His goal is to cut down to 6 cups per week. If he starts with 24 cups the first week, cuts down to 21 cups the second week, and drops to 18 cups the third week, how many weeks will it take him to reach his goal?
5. Taylor checked out a book from the library and it is now 5 days late. The late fee is 10 cents per day. How much is the fine?
6. How many hours will a car traveling at 75 miles per hour take to catch up to a car traveling at 55 miles per hour if the slower car starts two hours before the faster car?
7. Grace starts biking at 12 miles per hour. One hour later, Dan starts biking at 15 miles per hour, following the same route. How long would it take him to catch up with Grace?
8. Lemuel wants to enclose a rectangular plot of land with a fence. He has 24 feet of fencing. What is the largest possible area that he could enclose with the fence?

Mixed Review

9. Determine if the relation is a function: $\{(2,6),(-9,0),(7,7),(3,5),(5,3)\}$.
10. Roy works construction during the summer and earns \$78 per job. Create a table relating the number of jobs he could work, j, and the total amount of money he can earn, m.
11. Graph the following order pairs: (4,4); (–5,6), (–1,–1), (–7,–9), (2,–5)
12. Evaluate the following expression: $-4(4z-x+5)$; use $x=-10$, and $z=-8$.
13. The area of a circle is given by the formula $A=\pi r^2$. Determine the area of a circle with radius 6 mm.
14. Louie bought 9 packs of gum at \$1.19 each. How much money did he spend?
15. Write the following without the multiplication symbol: $16\times\frac{1}{8}c$.

13.3 Problem-Solving Strategies: Guess and Check and Work Backwards

This lesson will expand your toolbox of problem-solving strategies to include **guess and check** and **work backwards**. Let's begin by reviewing the four-step problem-solving plan.

Step 1: Understand the problem.

Step 2: Devise a plan – Translate.

Step 3: Carry out the plan – Solve.

Step 4: Look – Check and Interpret.

Develop and Use the Strategy: Guess and Check

The strategy for the "guess and check" method is to guess a solution and use that guess in the problem to see if you get the correct answer. If the answer is too big or too small, then make another guess that will get you closer to the goal. You continue guessing until you arrive at the correct solution. The process might sound like a long one; however, the guessing process will often lead you to patterns that you can use to make better guesses along the way.

Here is an example of how this strategy is used in practice.

Example 1: *Nadia takes a ribbon that is 48 inches long and cuts it in two pieces. One piece is three times as long as the other. How long is each piece?*

Solution: We need to find two numbers that add to 48. One number is three times the other number.

Guess	5 and 15	the sum is $5+15=20$	which is too small
Guess bigger numbers	6 and 18	the sum is $6+18=24$	which is too small

However, you can see that the previous answer is exactly half of 48.

Multiply 6 and 18 by two.

Our next guess is	12 and 36	the sum is $12+36=48$	This is correct.

Develop and Use the Strategy: Work Backwards

The "work backwards" method works well for problems in which a series of operations is applied to an unknown quantity and you are given the resulting value. The strategy in these problems is to start with the result and apply the operations in reverse order until you find the unknown. Let's see how this method works by solving the following problem.

Example 2: *Anne has a certain amount of money in her bank account on Friday morning. During the day she writes a check for \$24.50, makes an ATM withdrawal of \$80, and deposits a check for \$235. At the end of the day, she sees that her balance is \$451.25. How much money did she have in the bank at the beginning of the day?*

Solution: We need to find the money in Anne's bank account at the beginning of the day on Friday. From the unknown amount, we subtract \$24.50 and \$80 and we add \$235. We end up with \$451.25. We need to start with the result and apply the operations in reverse.

Start with \$451.25. Subtract \$235, add \$80, and then add \$24.50.

$$451.25 - 235 + 80 + 24.50 = 320.75$$

Anne had \$320.75 in her account at the beginning of the day on Friday.

Plan and Compare Alternative Approaches to Solving Problems

Most word problems can be solved in more than one way. Often one method is more straightforward than others. In this section, you will see how different problem-solving approaches compare for solving different kinds of problems.

Example 3: *Nadia's father is 36. He is 16 years older than four times Nadia's age. How old is Nadia?*

Solution: This problem can be solved with either of the strategies you learned in this section. Let's solve the problem using both strategies.

Guess and Check Method:

We need to find Nadia's age.

We know that her father is 16 years older than four times her age, or $4\times$ (Nadia's age) + 16.

We know her father is 36 years old.

Work Backwards Method:

Nadia's father is 36 years old.

To get from Nadia's age to her father's age, we multiply Nadia's age by four and add 16.

Working backward means we start with the father's age, subtract 16, and divide by 4.

Solve Real-World Problems Using Selected Strategies as Part of a Plan

Example 4: *Hana rents a car for a day. Her car rental company charges \$50 per day and \$0.40 per mile. Peter rents a car from a different company that charges \$70 per day and \$0.30 per mile. How many miles do they have to drive before Hana and Peter pay the same price for the rental for the same number of miles?*

Solution: Hana's total cost is \$50 plus \$0.40 times the number of miles.

Peter's total cost is \$70 plus \$0.30 times the number of miles.

Guess the number of miles and use this guess to calculate Hana's and Peter's total cost.

Keep guessing until their total cost is the same.

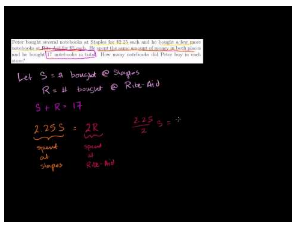
Guess	50 *miles*	
Check	$\$50 + \$0.40(50) = \$70$	$\$70 + \$0.30(50) = \$85$
Guess	60 *miles*	
Check	$\$50 + \$0.40(60) = \$74$	$\$70 + \$0.30(60) = \$88$

Notice that for an increase of 10 miles, the difference between total costs fell from \$15 to \$14. To get the difference to zero, we should try increasing the mileage by 140 miles.

Guess	200 *miles*		
Check	$\$50 + \$0.40(200) = \$130$	$\$70 + \$0.30(200) = \$130$	correct

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. CK-12 Basic Algebra:ProblemSolvingWord Problems 2 (12:20)



MEDIA

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1. Finish the problem we started in Example 3.
2. Nadia is at home and Peter is at school, which is 6 miles away from home. They start traveling toward each other at the same time. Nadia is walking at 3.5 miles per hour and Peter is skateboarding at 6 miles per hour. When will they meet and how far from home is their meeting place?
3. Peter bought several notebooks at Staples for \$2.25 each and he bought a few more notebooks at Rite-Aid for \$2 each. He spent the same amount of money in both places and he bought 17 notebooks in total. How many notebooks did Peter buy in each store?
4. Andrew took a handful of change out of his pocket and noticed that he was holding only dimes and quarters in his hand. He counted that he had 22 coins that amounted to \$4. How many quarters and how many dimes does Andrew have?
5. Anne wants to put a fence around her rose bed that is one and a half times as long as it is wide. She uses 50 feet of fencing. What are the dimensions of the garden?
6. Peter is outside looking at the pigs and chickens in the yard. Nadia is indoors and cannot see the animals. Peter gives her a puzzle. He tells her that he counts 13 heads and 36 feet and asks her how many pigs and how many chickens are in the yard. Help Nadia find the answer.
7. Andrew invests \$8000 in two types of accounts: a savings account that pays 5.25% interest per year and a more risky account that pays 9% interest per year. At the end of the year, he has \$450 in interest from the two accounts. Find the amount of money invested in each account.
8. There is a bowl of candy sitting on our kitchen table. This morning Nadia takes one-sixth of the candy. Later that morning Peter takes one-fourth of the candy that's left. This afternoon, Andrew takes one-fifth of what's left in the bowl and finally Anne takes one-third of what is left in the bowl. If there are 16 candies left in the bowl at the end of the day, how much candy was there at the beginning of the day?
9. Nadia can completely mow the lawn by herself in 30 minutes. Peter can completely mow the lawn by himself in 45 minutes. How long does it take both of them to mow the lawn together?

13.4 Problem-Solving Strategies - Using Graphs to solve a problem

In this section, we will look at a few examples of linear relationships that occur in real-world problems and see how we can solve them using grpahs. Remember back to our Problem Solving Plan:

1. **Understand the problem**
2. **Devise a plan - Translate**
3. **Carry out the plan - Solve**
4. **Look - Check and Interpret**

Example 1

A cell phone company is offering its costumers the following deal. You can buy a new cell phone for $60 and pay a monthly flat rate of $40 per month for unlimited calls. How much money will this deal cost you after 9 months?

Solution

Let's follow the problem solving plan.

Step 1: The cell phone costs $60, the calling plan costs $40 per month

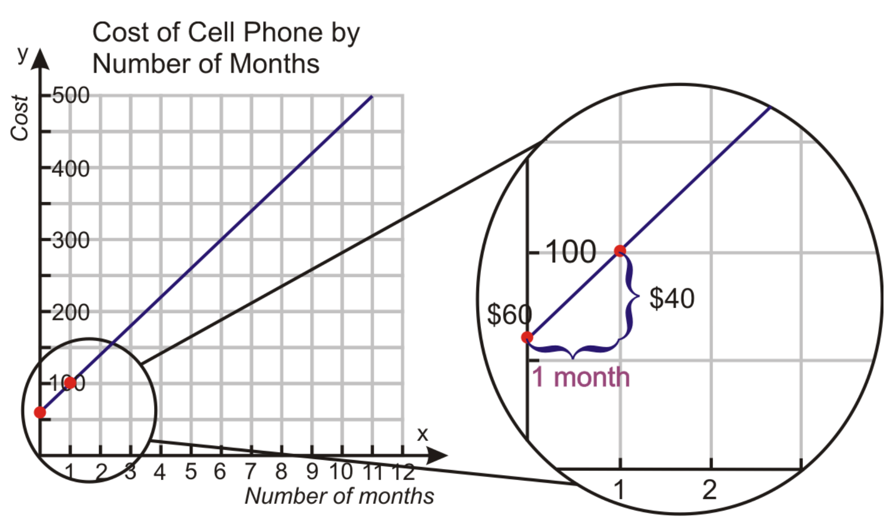
Let $x =$ number of months
Let $y =$ total cost in dollars

Step 2: Let's solve this problem by making a graph that shows the number of months on the horizontal axis and the cost on the vertical axis.

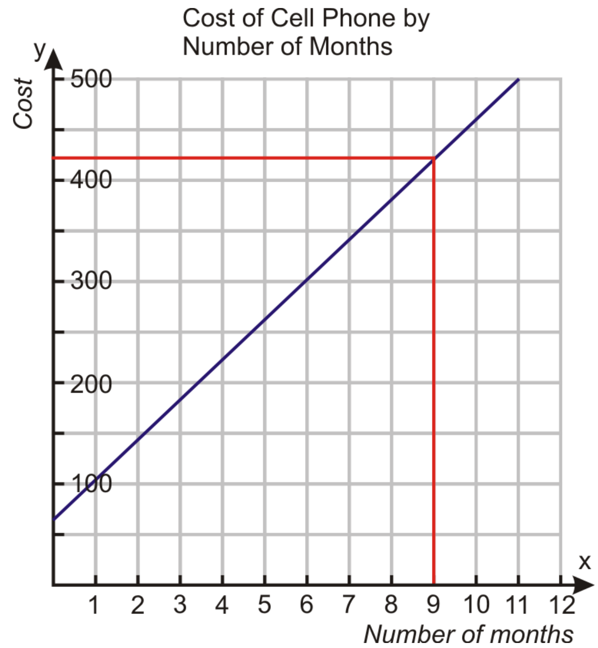
Since you pay $60 for the phone when you get the phone, then the $y-$intercept is (0, 60).

You pay $40 for each month, so the cost rises by $40 for one month, so the slope = 40.

We can graph this line using the slope-intercept method.



Step 3: The question was: *"How much will this deal cost after 9 months?"*



We can now read the answer from the graph. We draw a vertical line from 9 months until it meets the graph, and then draw a horizontal line until it meets the vertical axis.

We see that after 9 months **you pay approximately** \$420.

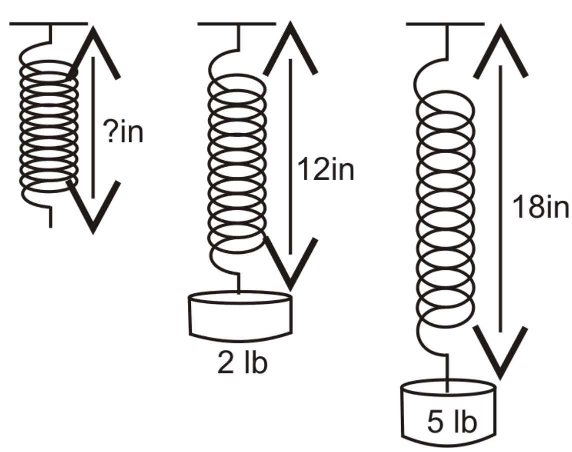
Step 4: To check if this is correct, let's think of the deal again. Originally, you pay \$60 and then \$40 for 9 months.

$$\text{Phone} = \$60$$
$$\text{Calling plan} = \$40 \times 9 = \$360$$
$$\text{Total cost} = \$420.$$

The answer checks out.

Example 2

A stretched spring has a length of 12 inches when a weight of 2 lbs is attached to the spring. The same spring has a length of 18 inches when a weight of 5 lbs is attached to the spring. It is known from physics that within certain weight limits, the function that describes how much a spring stretches with different weights is a linear function. What is the length of the spring when no weights are attached?



Solution

Let's apply problem solving techniques to our problem.

Step 1:

We know: the length of the spring = 12 inches when weight = 2 lbs.

the length of the spring = 18 inches when weight = 5 lbs.

We want: the length of the spring when the weight = 0 lbs.

Let $x =$ the weight attached to the spring.

Let $y =$ the length of the spring

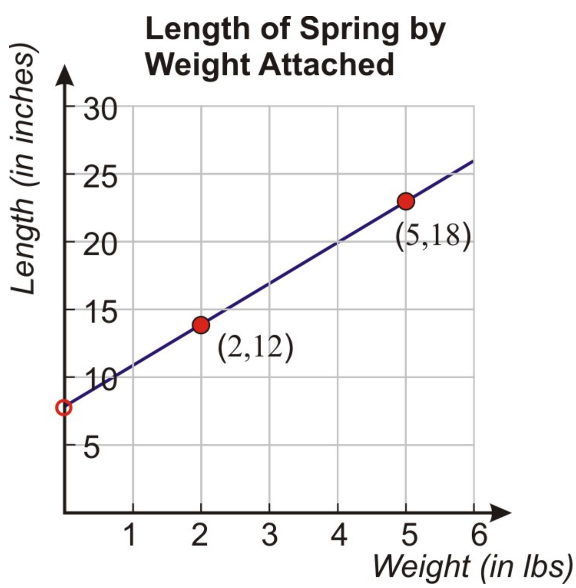
Step 2

Let's solve this problem by making a graph that shows the weight on the horizontal axis and the length of the spring on the vertical axis.

We have two points we can graph.

When the weight is 2 lbs, the length of the spring is 12 inches. This gives point (2, 12).

When the weight is 5 lbs, the length of the spring is 18 inches. This gives point (5, 18).



If we join these two points by a line and extend it in both directions we get the relationship between weight and length of the spring.

Step 3

The question was: *"What is the length of the spring when no weights are attached?"*

We can answer this question by reading the graph we just made. When there is no weight on the spring, the x value equals to zero, so we are just looking for the $y-$intercept of the graph. Looking at the graph we see that the $y-$intercept is **approximately 8 inches.**

Step 4

To check if this correct, let's think of the problem again.

You can see that the length of the spring goes up by 6 inches when the weight is increased by 3 lbs, so the slope of the line is $\frac{6 \ inches}{3 \ lbs} = 2 \ inches/lb$.

To find the length of the spring when there is no weight attached, we look at the spring when there are 2 lbs attached. For each pound we take off, the spring will shorten by 2 inches. Since we take off 2 lbs, the spring will be shorter by 4 inches. So, the length of the spring with no weights is $12\ inches\ - 4\ inches = 8\ inches$.

The answer checks out.

Example 3

Christine took one hour to read 22 pages of *Harry Potter and the Order of the Phoenix*. She has 100 pages left to read in order to finish the book. Assuming that she reads at a constant rate of pages per hour, how much time should she expect to spend reading in order to finish the book?

Solution: Let's apply the problem solving techniques:

Step 1: We know that it takes Christine takes 1 hour to read 22 pages.

We want to know how much time it takes her to read 100 pages.

Let $x =$ the time expressed in hours.

Let $y =$ the number of pages.

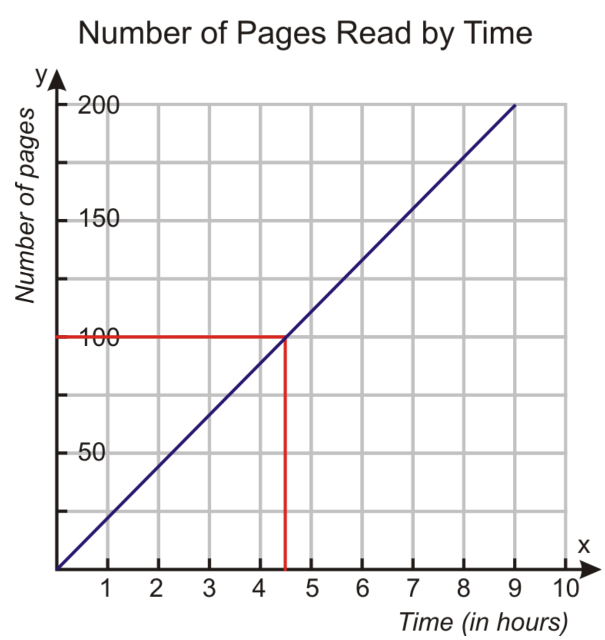
Step 2: Let's solve this problem by making a graph that shows the number of hours spent reading on the horizontal axis and the number of pages on the vertical axis.

We have two points we can graph.

Christine takes one hour to read 22 pages. This gives point (1, 22).

A second point is not given but we know that Christine takes 0 hours to read 0 pages. This gives the point (0, 0).

If we join these two points by a line and extend it in both directions we get the relationship between the amount of time spent reading and the number of pages read.



Step 3: The question was: *"How much time should Christine expect to spend reading 100 pages?"*

We find the answer from reading the graph - we draw a horizontal line from 100 pages until it meets the graph and then we draw the vertical until it meets the horizontal axis. We see that it takes **approximately 4.5 hours** to read the remaining 100 pages.

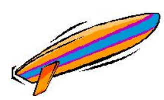
Step 4: To check if this correct, let's think of the problem again.

We know that Christine reads 22 pages per hour. This is the slope of the line or the rate at which she is reading. To find how many hours it takes her to read 100 pages, we divide the number of pages by the rate. In this case, $\frac{100 \text{ pages}}{22 \text{ pages per hour}} = 4.54$ hours. This is very close to what we gathered from reading the graph.

The answer checks out.

Example 4

Aatif wants to buy a surfboard that costs \$249. He was given a birthday present of \$50 and he has a summer job that pays him \$6.50 per hour. To be able to buy the surfboard, how many hours does he need to work?



Solution

Let's apply the problem solving techniques.

Step 1

We know - Surfboard costs \$249.

He has \$50.

His job pays \$6.50 per hour.

We want - How many hours does Aatif need to work to buy the surfboard?

Let $x =$ the time expressed in hours

Let $y =$ Aatif's earnings

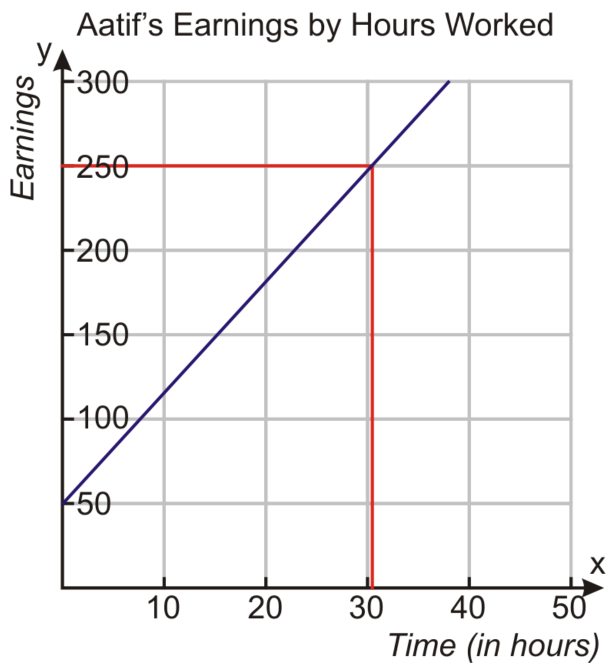
Step 2

Let's solve this problem by making a graph that shows the number of hours spent working on the horizontal axis and Aatif's earnings on the vertical axis.

Peter has \$50 at the beginning. This is the $y-$intercept of (0, 50).

He earns \$6.50 per hour. This is the slope of the line.

We can graph this line using the slope-intercept method. We graph the $y-$intercept of (0, 50) and we know that for each unit in the horizontal direction the line rises by 6.5 units in the vertical direction. Here is the line that describes this situation.



Step 3

The question was *"How many hours does Aatif need to work in order to buy the surfboard?"*

We find the answer from reading the graph. Since the surfboard costs \$249, we draw a horizontal line from \$249 on the vertical axis until it meets the graph and then we draw a vertical line downwards until it meets the horizontal axis. We see that it takes**approximately 31 hours** to earn the money.

Step 4

To check if this correct, let's think of the problem again.

We know that Aatif has \$50 and needs \$249 to buy the surfboard. So, he needs to earn $\$249 - \$50 = \$199$ from his job.

His job pays \$6.50 per hour. To find how many hours he need to work we divide $\frac{\$199}{\$6.50 \text{ per hour}} = 30.6$ hours. This is very close to the result we obtained from reading the graph.

The answer checks out.

Review Questions

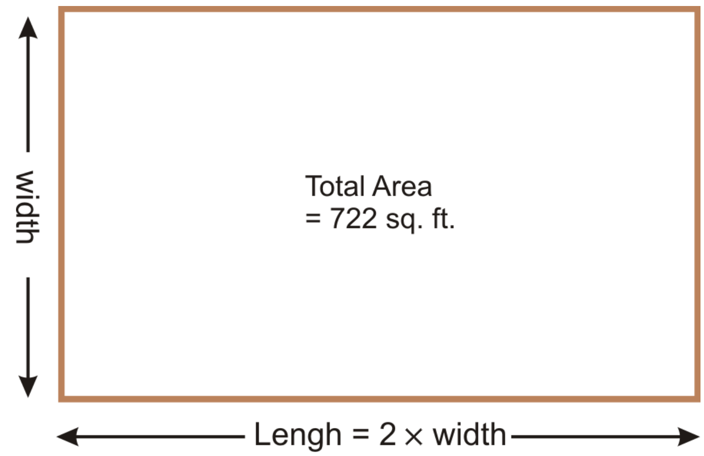
Solve the following problems by making a graph and reading a graph.

1. A gym is offering a deal to new members. Customers can sign up by paying a registration fee of \$200 and a monthly fee of \$39. How much will this membership cost a member by the end of the year?
2. A candle is burning at a linear rate. The candle measures five inches two minutes after it was lit. It measures three inches eight minutes after it was lit. What was the original length of the candle?
3. Tali is trying to find the width of a page of his telephone book. In order to do this, he takes a measurement and finds out that 550 pages measures 1.25 inches. What is the width of one page of the phone book?
4. Bobby and Petra are running a lemonade stand and they charge 45 cents for each glass of lemonade. In order to break even they must make \$25. How many glasses of lemonade must they sell to break even?

13.5 Problem-Solving Strategies: Use a Formula

Some problems are easily solved by applying a formula, such as The Percent Equation or the area of a circle. In this lesson, you will include using formulas in your toolbox of problem-solving strategies.

An architect is designing a room that is going to be twice as long as it is wide. The total square footage of the room is going to be 722square feet. What are the dimensions in feet of the room?



This situation applies very well to a formula. The formula for the area of a rectangle is: $A = l(w)$, *where* $l =$ *length and* $w =$ *width*. From the situation, we know the length is twice as long as the width. Translating this into an algebraic equation, we get:

$$A = (2w)w$$

Simplifying the equation: $A = 2w^2$

Substituting the known value for A: $722 = 2w^2$

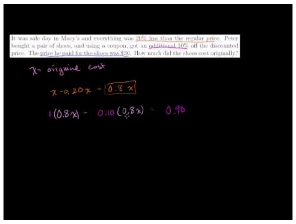
$$\begin{aligned} 2x^2 &= 722 && \text{Divide both sides by 2.} \\ x^2 &= 361 && \text{Take the square root of both sides.} \\ x &= \sqrt{361} = 19 \\ 2x &= 2 \times 19 = 38 \\ x &= 19 \end{aligned}$$

The width is 19 feet and the length is 38 feet.

Practice Set

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the

practice exercise listed in the following exercise set. However, the practice exercise is the same in both. CK-12 Basic Algebra:Word Problem Solving 3 (11:06)



MEDIA

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1. Patricia is building a sandbox for her daughter. It's going to be five feet wide and eight feet long. She wants the height of the sandbox to be four inches above the height of the sand. She has 30 cubic feet of sand. How high should the sandbox be?
2. A 500-sheet stack of copy paper is 1.75 inches high. The paper tray on a commercial copy machine holds a two foot high stack of paper. Approximately how many sheets is this?
3. It was sale day at Macy's and everything was 20% less than the regular price. Peter bought a pair of shoes, and using a coupon, got an additional 10% off the discounted price. The price he paid for the shoes was $36. How much did the shoes cost originally?
4. Peter is planning to show a video file to the school at graduation, but he's worried that the distance that the audience sits from the speakers will cause the sound and the picture to be out of sync. If the audience sits 20 meters from the speakers, what is the delay between the picture and the sound? (The speed of sound in air is 340 meters per second).
5. Rosa has saved all year and wishes to spend the money she has on new clothes and a vacation. She will spend 30% more on the vacation than on clothes. If she saved $1000 in total, how much money (to the nearest whole dollar) can she spend on the vacation?
6. On a DVD, data is stored between a radius of 2.3 cm and 5.7 cm. Calculate the total area available for data storage in square cm.
7. If a Blu-ray ^{TM} DVD stores 25 gigabytes (GB), what is the ***storage density***, in GB per square cm ?
8. The volume of a cone is given by the formula $Volume = \frac{\pi r^2(h)}{3}$, *where* $r =$ *radius*, *and* $h =$ *height of cone*. Determine the amount of liquid a paper cone can hold with a 1.5 inches diameter and a 5 inches height.
9. Consider the conversion $1\ meter = 39.37\ inches$. How many inches are in a kilometer? (Hint: A kilometer is equal to 1,000 meters)
10. Yanni's motorcycle travels $108\ miles/hour$. $1\ mph = 0.44704\ meters/second$. How many meters did Yanni travel in 45 seconds?
11. The area of a rectangle is given by the formula $A = l(w)$. A rectangle has an area of 132 square centimeters and a length of 11 centimeters. What is the perimeter of the rectangle?
12. The surface area of a cube is given by the formula: $SurfaceArea = 6x^2$, *where* $x =$ *side of the cube.* Determine the surface area of a die with a 1 inch side length.

Lesson Summary

The four steps of the **problem solving plan** when using graphs are:

1. **Understand the Problem**
2. **Devise a Plan—Translate:** Make a graph.
3. **Carry Out the Plan—Solve:** Use the graph to answer the question asked.
4. **Look—Check and Interpret**

Problem Set

Solve the following problems by making a graph and reading it.

1. A gym is offering a deal to new members. Customers can sign up by paying a registration fee of \$200 and a monthly fee of \$39.
   (a) How much will this membership cost a member by the end of the year?
   (b) The old membership rate was \$49 a month with a registration fee of \$100. How much more would a year's membership cost at that rate?
   (c) **Bonus:** For what number of months would the two membership rates be the same?
2. A candle is burning at a linear rate. The candle measures five inches two minutes after it was lit. It measures three inches eight minutes after it was lit.
   (a) What was the original length of the candle?
   (b) How long will it take to burn down to a half-inch stub?
   (c) Six half-inch stubs of candle can be melted together to make a new candle measuring $2\frac{5}{6}$ inches (a little wax gets lost in the process). How many stubs would it take to make three candles the size of the original candle?
3. A dipped candle is made by taking a wick and dipping it repeatedly in melted wax. The candle gets a little bit thicker with each added layer of wax. After it has been dipped three times, the candle is 6.5 mm thick. After it has been dipped six times, it is 11 mm thick.
   (a) How thick is the wick before the wax is added?
   (b) How many times does the wick need to be dipped to create a candle 2 cm thick?
4. Tali is trying to find the thickness of a page of his telephone book. In order to do this, he takes a measurement and finds out that 55 pages measures $\frac{1}{8}$ *inch*. What is the thickness of one page of the phone book?
5. Bobby and Petra are running a lemonade stand and they charge 45 cents for each glass of lemonade. In order to break even they must make \$25.
   (a) How many glasses of lemonade must they sell to break even?
   (b) When they've sold \$18 worth of lemonade, they realize that they only have enough lemons left to make 10 more glasses. To break even now, they'll need to sell those last 10 glasses at a higher price. What does the new price need to be?
6. Dale is making cookies using a recipe that calls for 2.5 cups of flour for two dozen cookies. How many cups of flour does he need to make five dozen cookies?
7. To buy a car, Jason makes a down payment of \$1500 and pays \$350 per month in installments.
   (a) How much money has Jason paid at the end of one year?
   (b) If the total cost of the car is \$8500, how long will it take Jason to finish paying it off?
   (c) The resale value of the car decreases by \$100 each month from the original purchase price. If Jason sells the car as soon as he finishes paying it off, how much will he get for it?

8. Anne transplants a rose seedling in her garden. She wants to track the growth of the rose so she measures its height every week. On the third week, she finds that the rose is 10 inches tall and on the eleventh week she finds that the rose is 14 inches tall. Assuming the rose grows linearly with time, what was the height of the rose when Anne planted it?
9. Ravi hangs from a giant spring whose length is 5 m. When his child Nimi hangs from the spring its length is 2 m. Ravi weighs 160 lbs and Nimi weighs 40 lbs. Write the equation for this problem in slope-intercept form. What should we expect the length of the spring to be when his wife Amardeep, who weighs 140 lbs, hangs from it?
10. Nadia is placing different weights on a spring and measuring the length of the stretched spring. She finds that for a 100 gram weight the length of the stretched spring is 20 cm and for a 300 gram weight the length of the stretched spring is 25 cm.
    (a) What is the unstretched length of the spring?
    (b) If the spring can only stretch to twice its unstretched length before it breaks, how much weight can it hold?
11. Andrew is a submarine commander. He decides to surface his submarine to periscope depth. It takes him 20 minutes to get from a depth of 400 feet to a depth of 50 feet.
    (a) What was the submarine's depth five minutes after it started surfacing?
    (b) How much longer would it take at that rate to get all the way to the surface?
12. Kiersta's phone has completely run out of battery power when she puts it on the charger. Ten minutes later, when the phone is 40% recharged, Kiersta's friend Danielle calls and Kiersta takes the phone off the charger to talk to her. When she hangs up 45 minutes later, her phone has 10% of its charge left. Then she gets another call from her friend Kwan.
    (a) How long can she spend talking to Kwan before the battery runs out again?
    (b) If she puts the phone back on the charger afterward, how long will it take to recharge completely?
13. Marji is painting a 75-foot fence. She starts applying the first coat of paint at 2 PM, and by 2:10 she has painted 30 feet of the fence. At 2:15, her husband, who paints about $\frac{2}{3}$ as fast as she does, comes to join her.
    (a) How much of the fence has Marji painted when her husband joins in?
    (b) When will they have painted the whole fence?
    (c) How long will it take them to apply the second coat of paint if they work together the whole time?